COMPUTATIONAL RELATIVITY: NUMERICAL AND ALGEBRAIC APPROACHES
REPORT OF WORKSHOP A4

Larry Smarr*

Departments of Astronomy and Physics
University of Illinois
Urbana, IL 61801, USA

The current use of computers in general relativity was reviewed at GR10. Today several groups have working codes which can calculate the coupled general relativity/hydrodynamics equations in two spatial dimensions plus time. These codes are compared and a critical review is made of the areas needing improvement. The potential uses of numerical relativity are discussed. In some situations the gravitational field can be chosen to be an analytically known solution, and hydrodynamics can be done in that background field. Regge calculus is still in its infancy numerically, but interesting new work has been done. Algebraic computing is becoming a mature technology, with the emphasis moving toward making these powerful software tools available to those relativists without previous computer training.

1. INTRODUCTION

There were two sessions of workshop A4 on numerical relativity at the Tenth International Conference on General Relativity and Gravitation . As chairman of that workshop, I am writing a summary of the state of the use of computers in general relativity. Rather than summarize the papers given (abstracts are published in Bertotti et al., 1983), I will attempt to outline the major efforts in this area and give a prospectus for the future. Because of the unending exponentiation in the power of computers, it is clear that relativity

B. Bertotti et al. (eds.), General Relativity and Gravitation, 163–183.
© 1984 by D. Reidel Publishing Company.

will be transformed just as so many other fields have been. However, relativity has been more resistant than other fields because of its proud heritage of analytic methods. What our field must recognize is that computational tools are not enemies of analytic tools. Indeed, each needs the other if continued progress is to be made. I believe that this marriage is occurring in each of the subfields I will survey: numerical relativity, relativistic hydrodynamics, Regge calculus, and algebraic computing. My review will emphasize the first topic, because there is a very good review talk on algebraic computing in this volume by Ian Cohen.

2. NUMERICAL RELATIVITY

Numerical relativity discretizes the spacetime continuum manifold into a spacetime lattice, the events of which are organized into a spacetime coordinate system. The differential Einstein equations have their derivatives replaced by a finite difference approximation. This transforms the differential field equations into a nonlinear coupled system of algebraic equations. The unknown variables are the components of the 4-metric tensor at the lattice points. As the lattice is refined to smaller scales, the solutions converge to the continuum solution of Einstein's differential equations. Note that, in contrast to the Regge calculus discussed in Section 4, the algebraic system of equations in numerical relativity are not the *exact* Einstein field equations on the chosen spacetime lattice.

In practice, this lattice of events is built up from an initial data hypersurface by constructing a family of hypersurfaces to the future of that data. There are two fundamentally different approaches to this construction: the spacelike and the null. In the first, the spacetime is built up from spacelike hypersurfaces starting on a Cauchy initial value surface (for a detailed definition see Eardley and Smarr, 1979). In the second, characteristic initial data is evolved using null hypersurfaces. Since there were no results on the null approach reported at GR10, I shall not discuss this important technique here, but refer the interested reader to papers by Stewart and Friedrich (1982) and Isaacson et al. (1983).

2.1. One Spatial Dimension Codes

There are well documented codes which can solve the Einstein/hydrodynamics equations in one spatial

dimension plus time. The time slicing is maximal for noncosmological spacetimes or spatially constant mean extrinsic curvature for cosmologies. The details of the equations and numerical techniques depend on the topology of the space slices. For examples see: SPHERICAL (Shapiro and Teukolsky, 1980), PLANAR (Centrella and Wilson, 1983a,b), and CYLINDRICAL (Piran, 1980). If the reader is not familiar with the 3+1 split of spacetime terminology which is used in this section, consult the review article of York (1979).

The use of these codes has been manyfold: 1) to test out differential (i.e. maximal slicing) or numerical (i.e. finite differencing schemes) techniques for use in multidimensional codes, 2) to elucidate the formation process of neutron stars and black holes, 3) to model gravitational radiation generation in the extreme of highly nonspherical collapse (e.g. cylindrical), 4) to calculate relativistic shock wave formation and interactions, and 5) to study highly anisotropic or inhomogeneous cosmology. Although these codes are physically highly idealized, they can yield extremely accurate solutions to fully relativistic problems. They can be used on small computers by researchers with a minimum of training. There is a great future in such codes which I will come back to in Section 2.4.

2.2. Two Spatial Dimension Codes

My emphasis will be on the multidimensional codes in which at least two spatial dimensions can be represented. These allow one to determine the spacetime representing nonspherical isolated gravitating systems with finite energy, which can radiate gravitational waves. The stress tensor source may vanish, as in black holes represented by Einstein-Rosen bridges or in pure gravitational wave spacetimes, or the stress tensor may be given by some matter field. In the latter case, the evolution equations for the matter must also be solved. Because the vacuum problem has been extensively reviewed elsewhere (Eppley, 1979; Smarr, 1979), I will concentrate only on the matter source codes.

There are two major groups (for shorthand the US and the Japanese groups) which have built Einstein/hydro codes and whose work was reviewed at GR10. These two groups have taken nearly opposite approaches which illustrate the wide variety of choice one has in the construction of numerical relativity codes. References to two groups work can be found in Evans (1984), Evans et al., (1984), Nakamura (1981, 1983).

Both codes assume a perfect fluid, typically with

a gamma law equation of state. Although both groups codes are axisymmetric, the US group (Evans, Smarr, and Wilson) does not allow for rotation, while the Japanese group (Nakamura, Sato, Maeda, et al.) does. Both groups use maximal slicing, with the Japanese group experimenting some with a hypergeometric slicing. The slicing condition sets the trace of the extrinsic curvature tensor equal to zero and forces the lapse function to satisfy an elliptic differential equation.

The selection of shift vectors leads to a dramatic difference between the form of the equations used by the two groups. The Japanese group sets the shift equal to zero, which, in their rotating case, means that all six of the 3-metric components are nonzero. This leads to a very large number of terms in any quantity, such as the 3-Ricci tensor, which appears in the Einstein equations. In contrast, the US group uses the "Wilson gauge" to set the shift vector. This gauge, in the nonrotating case, requires only two nonzero 3-metric components. These can be naturally chosen to be the overall 3-metric conformal factor and the Brill (1959) wave function q. Thus, in a spherical spacetime the radial coordinate is just the conformal Schwarzschild coordinate (with q=0).In a pure gravitational wave spacetime, the 3-metric is exactly that of the Brill wave on each spatial slice. This approach greatly aids the physical interpretation of the numerical results. As a consequence of their gauge choice, the Japanese do not have to solve for the shift vector, while the US group must solve a vector elliptic equation for the shift vector on each time slice.

Since the Einstein evolution equations are constrained, there is great flexibility in which set of equations one uses to solve for the components of the 3-metric (see review by Piran, 1980). The Japanese use a totally unconstrained evolution. This means that they solve six coupled pairs of hyperbolic equations for each of the 3-metric components and their corresponding extrinsic curvature components. The constraint equations are then evaluated on each time slice as a measure of deviation of the evolved data from the constrained path. The particular form of the evolution equations that they use is called "(2+1)+1" and is based on Geroch's (1971) three dimensional formalism for spacetimes with one (axial) Killing vector. The US group, in contrast, uses the full York (1979) formalism to turn the constraint equations into one scalar and one vector elliptic equation. These are solved on each time slice, leaving only one hyperbolic pair of evolution equations to be solved for. Thus, they perform

a fully constrained evolution.

In summary, the current Japanese code follows directly from approach of a zero shift, totally unconstrained evolution,used by Eppley (1979) and Smarr (1979) in the two black hole collision problem.There is no attempt in this method to isolate the gravitational wave degrees of freedom in the form of the 3-metric. The US group follows instead from the opposite approach, pioneered by Wilson (1979), in which one tries to simplify the form of the 3-metric as much as possible from the outset. In particular, the method then naturally provides metric variables for the constraint equations to constrain and wave variables for the evolution equations to evolve. Which method is superior remains to be shown, I believe. There still is a large amount of work to do in which the two methods are tried on the same problem with identical sized grids. Hopefully, cooperation between these two groups in the future will allow these comparisons.

In addition to the differences between the gauge and the mix of evolution/constraint equations used by the two groups, there is also a major difference between the numerical aspects of the two programs. The Japanese group has used a cylindrical coordinate system with a coarse spatial grid of 28×28 zones in the (R,Z) direction. This does not allow adequate zoning in the radial direction to determine the gravitational field in any detail, particularly gravitational radiation in black hole formation problems. Also, their finite differencing is first order (donor cell hydro, Lax-Friedrich for Einstein equations) which is very diffusive on a coarse grid. Realizing the limitations of their current numerical scheme, the Japanese group does not attempt to determine gravitational radiation information, but instead uses the code in a survey mode to determine the final states of gravitational collapse for a wide range of initial data. One should note that it is very easy for their group to greatly improve the fineness of their grid zoning and the accuracy of their numerical algorithm for solving the evolution equations, while not having to change the form of their differential equations at all. After these improvements are made, we will have a much better idea of the utility of their method.

The US group uses spherical coordinates and a much finer radial zoning (100 radial grid zones). The hydro differencing is closer to second order (see Wilson, 1979) with careful attention given to expansions of all functions near the origin (Evans et al., 1984). This allows the code to be used in a higher accuracy

mode to study particular problems in great detail, including gravitational radiation generation and propagation. Unfortunately, because of the large number of elliptic equations which must be solved for on each time slice, the code takes hours on a supercomputer for each spacetime evolved. Thus, it currently has not been used in survey mode to determine which problems are of most interest. Furthermore, it can not be used to calculate rotating problems. Clearly, what is needed is a combination of coarse survey runs to sketch out parameter space followed by the high accuracy runs on those spacetime which promise to be the most interesting.

2.3. The Crucial Role of Resolution

The issue of accuracy and resolution is a central and crucial one. The computation of gravitational radiation emission can be in error by a large factor, if one uses a coarse grid and low order hydro finite differencing. To illustrate the concerns see Figure 1. This shows a typical moment in a nonspherical gravitational collapse to a neutron star. Note that the core of the star has bounced along the z-axis and is rebounding outward, while most of the star in the equatorial plane is still collapsing inward. Consider also that the star is very inhomogeneous in density. This indicates that resolving the shape of the density contours and therefore the quadrupole moment of the star requires high spatial resolution. But since the luminosity in gravitational waves goes as $(\Delta Q)^2/(\Delta t)^6$, the time resolution must also be quite accurate. For instance, if the true time scale over which the quadrupole moment is reversed from oblate to prolate is a factor of two shorter than is resolved by the code, then the gravitational luminosity can be off by a factor of 64!

Even worse, Haugan et al. (1982) have shown that the relative phase of motion of each piece of matter in the star is crucial to the determination of the correct gravitational wave form. In their analysis, the coherent wave form has been built up from a lot of local oscillators. Destructive phase cancellation of the generated gravitational radiation can occur because of the light travel time differences across the star. Again, a poorly resolved matter flow will not have a hope of giving accurate enough relative phase information to ensure a correct final wave form.

Finally, almost every nonspherical gravitational collapse produces a curved shock front. As this shock

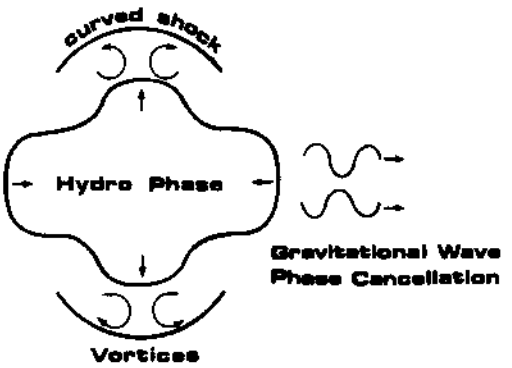


Fig. 1. A schematic diagram of a typical moment in the gravitational collapse of an oblate star to form a neutron star. The z-axis of symmetry is vertical. The core has just bounced and is rebounding along the z-axis. However, the collapse continues over most of the core. Outgoing curved shock fronts are generating vortices behind them. The outgoing gravitational waves are experiencing destructive cancellation because of light travel time differences across the core.

front propagates outward it reverses the direction of flow of the infalling matter. This means shocks can cause a very rapid change in the quadrupole moment and therefore a high gravitational radiation luminosity. Furthermore, curved shocks generate large scale vortices in the region behind the shock fronts. These rotational motions may have a critical effect on the emissivity in higher multipole moments. The correct resolution of the shape and strength of these shock fronts requires many grid zones. As an example of how a fine grid resolves shock fronts which are missed by a coarse grid, see Smarr and Hawley (1983).

In summary, while a coarse grid will allow one to calculate the evolution of initial data, the resulting approximation to the true spacetime may be very far off in both the energy radiated and the shape of the wave form produced.

There are even more difficulties when a black hole forms during the evolution. Figure 2 illustrates these new problems. The maximal slicing allows the time slices to penetrate inside the hole, while

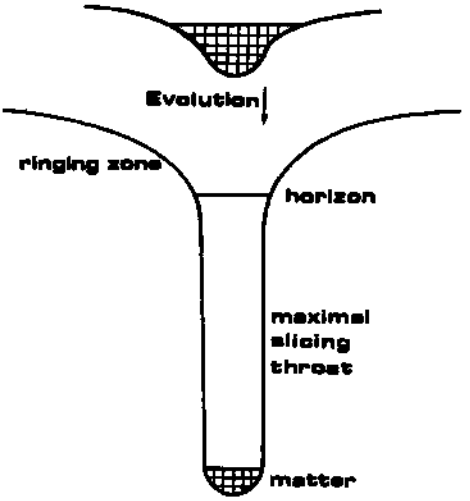


Fig. 2. Two phases of the evolution of the gravitational collapse of a star to form a black hole. The embedding diagram of the equatorial plane is shown. Maximal time slicing produces a long cylindrical throat with the matter at the bottom. The entire space from outside of the event horizon to about 15M is the wave generation or "ringing" zone.

avoiding the singularity to the future (see; e.g. Eardley and Smarr, 1979). The price one pays for this convenience is that the matter ends up at the bottom of a very long cylinder. The proper radius of this cylinder is 3M/2, while the length of the cylinder steadily increases with time (Eastabrook et al.,1973). The event horizon must be located somewhere above the cylinder. The location of either the true event horizon or the apparent horizon can only be determined numerically. From the event horizon out to about 15M, the gravitational radiation is generated in the "ringing zone" of the black hole. From perturbation theory (Press, 1971), we know that the wavelength of the radiation emerging from the hole will be 15-20M. This means we must locate our outer numerical boundary at least at 100M, if we are to follow the generation and propagation of a full wavetrain.

The numerical difficulties which are associated

with these physical and mathematical properties of black holes are extreme. The enormous stretching, during the evolution, of the proper distance from the center of the star to the outer grid point at 100M, means a very large number of radial grid points are going to be needed. These zones are "wasted" in the early phases of the calculation, but must be "held-in--reserve" for accurately resolving the final stretched configuration. Since the waves are redshifted as they emerge from the hole, there must be a higher density of grid points in the near hole region to resolve the shorter wavelength waves as they are generated. Similar remarks apply if one is going to accurately resolve the location, shape, area, and dynamics of the event horizon. It may be that sophisticated rezoning techniques will be needed if efficient and accurate black hole formation codes are to be created.

2.4. Future Numerical Relativity Codes

In many cases, Nature will not restrict herself to axisymmetry. As Miller (1983) emphasized in his presentation at GR10, many rotating initial data will result in collapse to a configuration with a Kerr parameter a/M>1. Almost surely, he argued, such a situation will lead to a nonaxisymmetric rotating bar which could be a copious producer of gravitational radiation. To model this four dimensional dynamic strong field spacetime is beyond the capability of any existing code.It is also beyond the capability of a Cray-1 supercomputer. Because all of the arguments on grid resolution made above apply to the third spatial dimension as well, it it clear that a fully 3-d code would require 50-100 times the storage and calculation time of today's 2-d codes. Fortunately, there is hope for attaining large increases in memory and speed in the next generation of computers, such as the Cray X-MP. These hardware increases coupled with continued algorithm improvements make accurate 3-d codes conceivable within the next five years.

In the meantime, calibration and higher resolution will be watchwords of 2-d codes. Now that there are several working codes, they must be turned into reliable tools for research in general relativity. All possible analytic or perturbation solutions, whether of flat spacetime or of relativistic objects, must be computed with the codes to get the finite difference approximations calibrated (see, e.g.Eppley, 1979). One should expect agreement to less than a percent everywhere. Horizon finders and gravitational wave form

calculators must be put through rigorous documented tests. The same spacetime should be calculated at finer and finer resolution to determine when all the physics has been reliably determined. Ideally, several groups using different codes should calculate the same problem. As an example of this calibration procedure applied to general relativistic hydrodynamics, see Hawley et al., (1983b,c).

The 1-d codes can be made essentially analytic in their accuracy. They should be rewritten as time implicit codes with adaptive grids (see e.g. Winkler and Norman, 1984). This not only allows for extremely high spacetime resolution, but it also allows for several time scales or length scales to be present in the problem. That will be a critical feature as more microphysics (detailed equations of state, electromagnetic radiation coupled to hydro, viscosity, magnetic fields, etc.) is added to the 1-d codes. Because of their smaller size, one can add enough physics to 1-d codes to be able to study realistic astrophysical situations. The study of realistic gravitational collapse or early universe phenomenology is in its infancy. These 1-d codes will be powerful tools in the development of our knowledge in the next few years.

2.5. The Uses of Numerical Relativity

With the advent of high resolution calibrated numerical relativity codes, many issues of principle in general relativity can be attacked. The tool is the only one which can solve for the general globally hyperbolic solution of Einstein equations (Eardley and Smarr, 1979). Our knowledge of such general dynamic strong field spacetimes is primitive at present. Detailed gravitational waveforms can be calculated from a wide variety of situations. For the first time we will have a glimpse of what gravitational wave astronomy may uncover (see Drever's contribution). Already straight line motion of strong field bodies has been calculated (Smarr, 1979; Wilson, 1979). These spacetimes need to be recalculated with accurate codes and compared in detail with the work on equations of motion reported here by Damour. The quadrupole moment formalism controversy (see Walker's contribution) may find useful input when gravitational radiation is calculated from a numerical spacetime by the several contending methods. Since strong field spacetimes can also be calculated by numerical relativity, we will find which of the weak field formalisms break down first as the gravity becomes strong. Finally, the issues of singular-

ity structure and cosmic censorship (see Christodoulou's contribution) will get many new examples from numerical relativity. By relaxing the maximal slicing condition, one can "sneak up" on spacelike singularities and perhaps determine if Cauchy horizons are being formed (Eardley and Smarr, 1979; Eardley, 1979).

Therefore, I conclude that numerical relativity is coming into its own. There is an enormous amount of work to do yet, but there seems to be no road blocks in the way. The field is currently manpower and computer resources limited, but the latter is changing for the better very rapidly. It seems likely that GR11 will see dramatic advances in numerical relativity.

3. RELATIVISTIC HYDRODYNAMICS

Another use of computers is to solve laws of physics in a given gravitational field. That is, the dynamics of the matter does not feed back through the Einstein equations to modify the spacetime curvature. This approximation needs to be justified by some physical reasoning in each separate instance. Fortunately, there are a large class of astrophysical situations where such an approximation is quite accurate. I will briefly mention a number of problems of this type which were reported on at GR10.

There is strong evidence that a supermassive (one hundred million solar mass) black hole exists at the center of many active galactic nuclei. Accretion onto this hole powers the enormous luminosity of quasars and the radio jets emerging from radio galaxies. Only a few solar masses a year need to accrete in order to reach quasar power levels. Therefore, it seems to be a good approximation to investigate the physics of matter in a fixed Kerr background.

Two competing methods were described for using computers to calculate the three dimensional problem of a star passing close to a black hole and being tidally distorted. Luminet and Marck (1983) reported a general relativistic improvement of a Newtonian model of Carter and Luminet. Here the star is assumed to maintain an ellipsoidal (3 unequal axes) shape and move on a geodesic around the hole. The relativistic equations for the tidal deviations and the bulk compression are computed along the trajectory. An alternate approach, reported by Bicknell and Gingold (1983) in session B1, uses a 3-d Newtonian smoothed particle hydro code to compute the full hydrodynamic behavior of the star. They find that the shape does not remain ellipsoidal and that

shocks play a crucial role in preventing high central compressions. Unfortunately, their 3-d resolution is not adequate to get a definitive answer and they neglect relativistic effects in the hydrodynamics. In spite of the defects of the two approaches, it is very impressive to see the beginnings of 3-d dynamic Newtonian hydro in black hole spacetimes being attacked today.

John Hawley, in collaboration with Smarr and Wilson (1983a),reported in session B1 on a fully general relativistic hydrodynamics code which can calculate rotating axisymmetric fluid flow around a Kerr black hole. Although the code cannot calculate 3-d hydro, it does not suffer from the drawbacks of leaving out relativistic effects or having low resolution. The code is used to study gas accreting in bulk onto a black hole. When pressure forces are operative, it is found that strong intersecting shocks can form well inside r=6M. These shocks can reverse the flow of gas into the hole and form a rotating fat disk orbiting the hole. Future uses of the code will be to study the general accretion physics in the near hole region. Note that such a code can become a subroutine in a numerical relativity code to solve the axisymmetric rotating Einstein equations.

Multidimensional hydro codes are typically Eulerian, that is they fix the grid in space and let the matter flow past. Unfortunately, it turns out that in such an approach it is difficult to model special relativistic flows accurately. This is because the special relativistic equations are horribly nonlinear in the velocity. Therefore, little is known about the dynamics of matter moving very near the spead of light. Lanza et al. (1983) reported on a 1-d Lagrangian code (based on the approach of May and White, 1967), which can accurately model these flows. It assumes no self gravity (Minkowski spacetime background) and a simple radiation or Synge equation of state. Very interesting results on the formation and damping of relativistic shock waves were obtained. This approach should lead to a high resolution 1-d numerical relativity code which can study how gravity alters the picture.

Finally, fully 3-d cosmological numerical simulations are being carried out by Centrella and Melott and Davis et al. Here the gravitational background is assumed to be Friedmann in the large, but the 3-d dynamics of fragmentation of the matter field is calculated using Newtonian physics. This problem is of crucial importance because of the growing evidence, reported here by Sato, that the universe's dynamics is

governed by dark matter. The nature of this dark matter (heavy neutrinos or some more exotic elementary particles) can be an input to the codes so that different fragmentation scenarios can be calculated and compared to observations. The numerical approach is not to model the dark matter by hydro, but rather as collisionless matter. The two groups have very different numerical approaches, which must be compared to determine which is more appropriate to model the physics of the dark matter.

4. REGGE CALCULUS

There were two very good papers delivered in this session on the Regge calculus. A review of the use of Regge calculus to obtain solutions of Einstein's equations was reported by Brewin (1983) and new techniques for solving the geodesic equations in a Regge spacetime were reported by Williams and Ellis (1983). Both projects require computers to numerically solve the resulting equations.

Regge calculus is often thought to be an alternate way to do numerical relativity. Actually, the two formalisms are very different. In Regge calculus the spacetime is divided up into 4-dimensional blocks *before* the Einstein equations are written down. The metric is assumed to be flat inside of these blocks. The unknown is not the metric tensor, as in numerical relativity, but instead a set of scalars that represent the proper lengths of the legs of the blocks. The gravitational action can be written down in terms of the properties of these legs and extremized *on the discrete spacetime*. Therefore, the resulting nonlinear algebraic system of equations are the *exact* Einstein field equations on that discrete spacetime. There is no need for a coordinate system to label events in the spacetime.

Only the simplest analytically known spacetimes have been calculated using this formalism: Friedmann, Tolman, and Kasner cosmologies and the static Schwarzschild black hole. In this regard, Regge calculus is where numerical relativity was over a decade ago. Part of the difficulty seems to be in choosing an appropriate set of 4-blocks. Brewin reported some advances on this issue in the context of the closed Friedmann model. If one demands that the blocks obey the same symmetry as the continuum space, then there are only three possible tetrahedra models which span the 3-sphere, N=5,16, and 600 (Collins and Williams,

1973). Brewin showed that the evolution of any such model has a finite endpoint some time before the final Big Crunch of the continuum solution. Furthermore, the radius versus time paths of the models are quite far from the analytic solution. Brewin proved that the first problem is generic to the Regge calculus, but the second problem can be relieved by softening the symmetry requirements. This procedure allowed him to subdivide each of the tetrahedra into twelve smaller tetrahedra. The N=7200 model was then quite close to the analytic model.

Williams and Ellis explained how one could use their method for tracing geodesics in a Regge spacetime. Within a flat block, the geodesic goes in a straight line. At the boundaries the geodesic suffers a "refraction" of a certain form. They studied light deflection, perihelion shifts, Thomas precession, and tidal forces in a Schwarzschild Regge geometry. Results were only close to correct if thousands of blocks were used, a remarkably similar finding to Brewin's. Furthermore, things could be very far off if they were not using the a priori knowledge of the Regge geometry. Again, it it not clear whether these difficulties are fundamental or only the result of poor resolution.

The major problem with Regge calculus is that no one seems to know what it is the answer to. It currently does not seem to be competitive with numerical relativity in constructing new solutions of Einstein equations. The fact that thousands of blocks must be used to get convergence to the true solutions, even in homogeneous or static spacetimes does not bode well for calculations of more complicated solutions. There has been no research on how matter shock waves will be treated, how gravitational waves can be resolved, what time slicings will work in black hole interiors, etc. As a result, one has no way to say today whether Regge calculus can reproduce the successes already achieved by numerical relativity. On the other hand, finite element methods, which Regge calculus seems related to, are enormously successful as a competitor to finite difference methods in hydrodynamics and other engineering simulations. Therefore, perhaps it is just a matter of more researchers working on the subject.

Certainly a formalism as elegant as the Regge calculus was meant as an answer to something important. Both Brewin and Ellis speculated that it might lie in the quantum gravity arena, where the spacetime continuum may need to be replaced by a discrete structure. If so then the computer may have a major contribution to make to that specialty of general relativity as

well.

5. ALGEBRAIC COMPUTING

The ability to manipulate long strings of symbolic expressions is the heart of general relativity theory. Although many of us pride ourselves for our capacity to fill reams of paper without making mistakes, I do not believe that such a capability should be confused with creative science. Therefore, it was encouraging to see how much progress has been made in the development of easy-to-use algebraic computing software. Since a detailed invited talk by Ian Cohen on the subject is included in the proceedings, and since I am not an expert on the merits of the competing systems, I will make only a few remarks on how our workshop appeared to a relativist eager to use such packages.

What an algebraic computing program does is use a rulebook for algebra, differentiation, integration, tensor calculations, etc. to take input from you and convert it to some other form you desire. It does all the same intermediate steps you would do, but very much faster and without error.It does not substitute in numerical values for variables, but rather deals in strings of those variables. An example would be for you to type in the algebraic form of some metric and it would give back the expression for the Riemann tensor components. It can also check relationships between functions. For instance, it can compare the different components of the Riemann tensor to decide the Petrov class of the spacetime.

One of the most exciting new uses of such packages was described by MacCallum (1983a) at GR10. It is a procedure for classifying all exact solutions of the Einstein equations by their invariants. The project will catalog all known solutions, enabling the user to type in a metric he is interested in and have the program tell him what solutions are isometric to his. The current version of this project is using the algebraic package SHEEP.

Given these general properties of any algebraic package, what makes one package better than another? The first question is whether you can run it on the type of computer that you have access to. Most of these packages are written in LISP, a common general purpose list manipulation language. Unfortunately, there are many dialects of LISP and which version your machine will run may not be the dialect a relativity package was written in. This is referred to as the portability

problem. Secondly, the different packages require varying amounts of memory to run and their speed can vary widely. It used to be that most discussions on the merits of different packages were centered on the memory requirements and speed. This does not seem to be as big an issue today because of great improvements in these properties by all the major packages.

The major issue today seems to be usability. We are seeing a shift from a small user base of experienced programmers to a much larger user base of relativists who are inexperienced in using computers. For this to happen, the authors of the package have to spend a lot of their time programming in "user friendliness". Examples of these features are: 1) ability to input and output in "textbook" notation, with familiar symbols, sub and superscripts, etc.; 2) interactive features where a user can decide at the keyboard what he wants next; 3) intelligent prompts in plain language to help the user through the calculation; and 4) online "help" packages so the user can teach himself how to use the package.

Equally important are clearly written user manuals with lots of worked examples. The support that the authors can give the user later is going to become more and more a deciding factor. If we are going to have hundreds of relativists using these packages, a one person support group is not going to be adequate. Thus, packages developed by a research group may become preferable.

The generality of the package has to be considered. Several general purpose packages which cover all types of physics and mathematics exist, such as MACSYMA, REDUCE, and SMP. Many relativists may never use large portions of these packages. However, they may pay for the extra generality in reduced speed and increased memory requirements. A more useful arrangement is to have greater modularity of the packages. A system would have a central "kernel" that contains the "symbol crunching" core of the package. Then in an overlying "shell" are modules that perform specific tasks when linked to the kernel. In this architecture, the user only has to load those modules that he needs for the calculation at hand. A nice step in this direction was the STENSOR package reported on by Hornfeldt of the Stockholm group. It can sit on top of either MACSYMA or SHEEP and manipulate tensor and spinor formulae in textbook conventions.

Given all the above considerations, it seems that the major contenders at this stage are SHEEP, CAMAL, REDUCE, and MACSYMA. A detailed comparison of these

and the other packages is given by Cohen in his invited talk. Of these SHEEP seems to be very compact and fast, well supported, easy to obtain, and interactive. It is written just for relativists and can run STENSOR on top of it. For examples of its use see MacCallum (1983b).

The final general trend I will comment on is the revolution in hardware which is occurring today. Until a few years ago, all these packages had to be run on mainframe computers (costing millions of dollars), of which there might be one per university. Often these had to be run in batch mode, so the turn around was very slow. Then the superminis (costing hundreds of thousands of dollars), such as the DEC VAX series, became widely available. Today many departments or active timesharing operating systems. The user can ractive timesharing operating systems. The user can work at a terminal with fairly fast turn around if there are only a few users on the machine. What is appearing this year are the first of the next generation machines called scientific work stations (costing tens of thousands of dollars). These condense a supermini to a desk top machine owned by a single user. Examples are APOLLO, SUN, and the just announced microVAX. So what we are seeing is an evolution in which many research scientists will do most of their research, writing papers, communicating with colleagues, etc. on one machine sitting on their desk. This is happening today and will be commonplace in five years.

It is clear that the relativity algebraic computing packages will have to follow this hardware trend. Indeed, a number of these packages are being ported to the intelligent work stations. However, the integration of these tools into the scientists working environment on the work station must also be accomplished. Since these work stations can also act as post processing graphics work stations for the numerical relativity codes described in Section 2, the relativist of five years from now may not make a major distinction between algebraic and numerical relativity. He or she will simply call it computational relativity.

ACKNOWLEDGEMENTS. I would like to thank all the participants in the workshop on numerical relativity at GR10 for their lively discussions. My review of numerical relativity was greatly helped by Charles Evans and James Wilson. Special thanks to Peter Barnes for lettering the Figures. Financial support was partially

provided by the NSF under grants PHY 80-01496 and PHY 83-08826

NOTE

* Alfred P.Sloan Fellow.

REFERENCES

Papers presented to the GR10 Conference and published in:

Bertotti, B., de Felice, F., and Pascolini, A. (eds): 1983, *10th International Conference on General Relativity and Gravitation. Contributed Papers*, Vols 1-2, CNR, Roma.

Brewin, L.: 1983, "Computer generated solutions using Regge calculus", p. 427.

Bicknell, G.V., and Gingold, R.A.: 1983, "Tidal detonation of stars by black holes", p. 690.

Hawley, J., Smarr, L., and Wilson, J.R.: 1983, "Numerical simulation of flat disks around black holes", p. 709.

Lanza, A., Miller, J.C., and Motta, S.: 1983, "Formation and damping of relativistic strong shocks", p. 436.

Luminet, J.P., and Marck, J.A.: 1983, "Tidal effects in Kerr geometry", p. 438.

MacCallum, M.A.H.: 1983a, "Proposed format for recording the invariant characterisation of exact solutions", p. 301.

Miller, J.C.: 1983, "Non-axisymmetric instabilities in gravitational collapse", p. 443.

Williams, R.M., and Ellis, G.F.R.: 1983, "Observational results from Regge calculus", p. 451.

Other references

Brill, D.R.: 1959, "On the Positive Definite Mass of

the Bondi-Weber-Wheeler Time-Symmetric Gravitational Waves", *Ann.Phys.*, 7, 466.

Centrella, J., and Wilson, J.R.: 1983a, "Planar numerical cosmology. I. The differential equations", *Astrophys.J.*, 273, 428.

Centrella, J., and Wilson, J.R.: 1983b, "Planar Cosmology. II. The difference equations and numerical tests", University of Illinois, Astronomy preprint 83-11.

Collins, P.A., and Williams, R.M.: 1973, "Dynamics of the Friedmann Universe Using Regge Calculus", *Phys. Rev.*, D7, 965.

Eardley, D.M.: 1979, "Global problems in numerical relativity", in L.Smarr (ed.), *Sources of Gravitational Radiation*, Cambridge University Press, Cambridge, pp. 127-138.

Eardley, D.M., and Smarr, L.: 1979, "Time functions in numerical relativity: marginally bound dust collapse", *Phys. Rev.*, D19, 2239.

Eppley, K.: 1979, "Pure gravitational waves", in L. Smarr (ed.), *Sources of Gravitational Radiation*, Cambridge University Press, Cambridge, pp.275-292.

Estabrook, F., Wahlquist, H., Christensen, S., DeWitt, B., Smarr, L., and Tsiang, E.: 1973, "Maximally Slicing a Black Hole", *Phys. Rev.*, D7, 2814.

Evans, C.: 1984, "A method for numerical simulation of gravitational collapse and gravitational radiation generation", in J.Centrella, R.Bowers, J.LeBlanc, and M. LeBlanc (eds), *Numerical Astrophysics: A Festschrift in Honor of James R. Wilson.*

Evans, C., Smarr, L., and Wilson, J.R.: 1984, "Numerical relativistic collapse and collisions with spatial time slices", in K.-H.A. Winkler, and M.L.Norman (eds), *Radiation Hydrodynamics*, Reidel, Dordrecht.

Geroch, R.: 1971, "A Method of Generating Solutions of Einstein's Equations", *J.Math. Phys.*, 12, 918.

Haugan, M.P., Shapiro, S.L., and Wasserman, I.: 1982, "The soppression of gravitational radiation from

finite-size stars falling into black holes", *Astrophys. J.*, 257, 283.

Hawley, J.F., Smarr, L., and Wilson, J.R.: 1983b, "A numerical study of nonspherical black hole accretion. I. Equations and test particles", University of Illinois Astronomy preprint 83-9, to appear in *Astrophys. J.*

Hawley, J.F., Smarr, L., and Wilson, J.R.: 1983c, "A numerical study of nonspherical black hole accretion. II. Finite differencing and code calibration", University of Illinois, Astronomy preprint, to appear in *Astrophys. J. Suppl.*

Isaacson, R.A., Wellings, J.S., and Winicour, J.: 1983, "Null cone computation of gravitational radiation", *J.Math. Phys.*, 24, 1824.

MacCallum, M.A.H.: 1983b, "Classifying metrics in theory and practice", in V.De Sabbata, and E.Schmutzer(eds), *Unified Field Theory of more than 4 Dimension Including Exact Solutions*, World Scientific Publishing, Singapore, p. 352.

May, M.M., and White, R.H.: 1967, "Stellar Dynamics and Gravitational Collapse", *Methods Comput. Phys.*, 7, 219.

Nakamura, T.: 1981, "General Relativistic Collapse of Axially Symmetric Stars Leading to the Formation of Rotating Black Holes", *Prog. Theor. Phys.*, 65, 1876.

Nakamura, T.: 1983, "General relativistic collapse of rotating stars", to appear in the proceedings of the 11th Texas Symposium on Relativistic Astrophysics.

Piran, T.: 1980, "Numerical codes for cylindrical general relativistic systems", *J.Comput. Phys.*, 35, 254.

Press, W.H.: 1971, "Long wave trains of gravitational waves from a vibrating black hole", *Astrophys.J. Lett.*, 170, L105.

Shapiro, S.L., and Teukolsky, S.A.: 1980, "Gravitational collapse to neutron stars and black holes: computers generation of spherical spacetimes",

Astrophys.J., 235, 199.

Smarr, L.: 1979, "Gauge condition, radiation formulae and the two black hole collision", in L.Smarr (ed.), *Sources of Gravitational Radiation*, Cambridge University Press, Cambridge, pp.245-274.

Smarr, L., and Hawley, J.F.: 1983, "General Relativistic Hydrodynamics and accretion physics: a numerical approach", to appear in the Proceedings of the Toulouse Workshop on Gravitational Collapse.

Stewart, J.M., and Friedrich, H.: 1982, "Numerical Relativity. I. The characteristic initial value problem", *Proc. Roy. Soc.*, A384, 427.

Wilson, J.R.: 1979, "A numerical method for relativistic hydrodynamics", in L.Smarr (ed.), *Sources of Gravitational Radiation*, Cambridge University Press, Cambridge, pp. 423-446.

Winkler, K.-H.A., and Norman, M.L.: 1984, "WH80's: Numerical Radiation Hydrodynamics", to appear in M.L.Norman, and K.-H.A.Winkler (eds),*Radiation Hydrodynamics*, Reidel, Dordrecht.

York, J.W., Jr.: 1979: "Kinematics and dynamics of general relativity", in L.Smarr (ed.), *Sources of Gravitational Radiation*, Cambridge University Press, Cambridge, pp. 83-126.