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General Relativistic Hydrodynamics: The Comoving, Eulerian, and Velocity Potential Formalisms

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Abstract

We compare the three major approaches to general relativistic hydrodynamics: comoving, Eulerian, or mixed Euler–Lagrange, and velocity potential. We show that the comoving gauge has three major disadvantages: (1) shocks cause discontinuities in the spacetime metric; (2) the formation of black holes can lead to time slicing singularities; (3) coordinate lines can become tangled in nonspherical flows. The mixed Euler–Lagrange scheme we use solves these three problems. We show how the comoving limit is obtained from our general equations. The circulation of a fluid flow in the presence of shocks is discussed and illustrated by numerical example. We suggest a possible relevance for this to supernova ejection mechanisms. Finally, the velocity potential formalism is discussed and compared to the comoving and Eulerian approaches. Taub's contributions are stressed throughout the discussion.

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I. Introduction

This paper arose from a series of discussions with Abe Taub over the last few years. Having been one of the strongest proponents of the use of comoving coordinates to describe relativistic hydrodynamics, he was interested in the possible limitations of this approach. We discuss that question here by using the ADM [1] kinematic analysis of spacetime coordinate systems [2]. We find three significant difficulties with the comoving gauge. First, shocks will cause discontinuities in either the lapse function or the shift vector. Second, if the comoving gauge is used to specify the time slicing, as in May and White's [3,28] or Pachner's [4] work, then the time slicing will become singular shortly ($\sim GMc^{-3}$) after the formation of a black hole. Third, if one is following nonspherical flow, then the growth of circulation behind shocks will tangle the spatial coordinates.

We show how removing the comoving restriction solves these three problems at the expense of having to integrate the full Eulerian hydrodynamics equations. The comoving limit is easily obtained from our equations. In the discussion of the third problem listed above, we review the circulation theorem in relativistic hydrodynamics following Taub's classic account [5]. A numerical example using the Eulerian approach shows clearly the generation of circulation after shocks form in a nonspherical collapse. We briefly discuss the relevance of this process for supernovae following Epstein [6] and Colgate's [7] recent remarks.

Finally, we review the relativistic velocity potential formalism introduced by Schutz [8]. We show how it related to both the comoving and Eulerian schemes and discuss its possible utility for numerical hydrodynamics.

II. Kinematics of Flows on Spacetime

As described in detail elsewhere [2], one can consider the coordinate system that labels spacetime events as a pair of flows on spacetime. These flows are smooth congruences of curves or trajectories. One flow is normal to the chosen foliation of time slices (t = constant). A unit tangent vector to this flow $\hat{n}^a$ defines the Eulerian observers in this coordinate system. The term Eulerian refers to the fact that these observers (unit timelike vectors) are at rest in the chosen time slices. The other flow is the spatial coordinate trajectories along which x^i ($i = 1, 2, 3$) are constant. The unit tangent vector for this flow is denoted $\hat{t}^a$ and it can be either timelike, null, or spacelike. The spacetime metric tensor g_{ab}, written in this coordinate system [1,2], defines the lapse function $\alpha \equiv (g^{tt})^{-1/2}$, the shift vector $\beta_i \equiv g_{ti}$, and the three metric $\gamma_{ij} = g_{ij}$.

In addition to these two flows, if the spacetime is not vacuum, there is the matter flow with unit timelike vector $\hat{u}^a$. The trajectories of this flow will not coincide with those of $\hat{n}^a$ or $\hat{t}^a$ in general. Thus, at each event in spacetime one has three unit vectors $(\hat{n}^a, \hat{t}^a, \hat{u}^a)$ that can be related by boosts. (We assume for now that $\hat{t}^a$ remains timelike; for the spacelike case, see Smarr and Wilson [9].) Since $\hat{n}^a$ is the Eulerian observer, it is natural to refer the boosts to it. In the coordinate system (t, x^i) defined by (α, β^i) above, these vectors are†

$$\hat{n}_a = (-\alpha, 0), \qquad \hat{n}^a = (\alpha^{-1}, -\beta^i/\alpha), \tag{1a}$$

$$\hat{t}_a = \Gamma_1(-\alpha + \beta_k\beta^k/\alpha, \beta_i/\alpha), \qquad \hat{t}^a = \Gamma_1(\alpha^{-1}, 0), \tag{1b}$$

$$\hat{u}_a = \Gamma_2(-\alpha + \beta_k v^k, v_i), \qquad \hat{u}^a = \Gamma_2(\alpha^{-1}, v^i - \beta^i/\alpha), \tag{1c}$$

$$-n\cdot t \equiv \Gamma_1 = (1 - \beta_k\beta^k/\alpha^2)^{-1/2}, \qquad -n\cdot u \equiv \Gamma_2 = (1 - v_k v^k)^{-1/2}. \tag{1d}$$

Thus, the boost velocity of $\hat{u}^a$ relative to $\hat{n}^a$ is v^i, while that of $\hat{t}^a$ relative to $\hat{n}^a$ is β^i/α. The two relativistic gamma factors are Γ_1 and Γ_2.

We can decompose the spacetime covariant derivative of each of these unit vector fields into its twist ω^{ab}, shear σ^{ab}, expansion θ, and acceleration ζ^a (for definitions, see Smarr and York Eqs. (4.2) and (4.3) [2]):

$$\nabla^a\hat{Z}^b = \omega^{ab} + \sigma^{ab} + \tfrac{1}{3}h^{ab}\theta - \hat{Z}^a\zeta^b. \tag{2a}$$

Here $\hat{Z}^b$ is an arbitrary unit timelike vector with its projection operator $\hat{h}^{ab}$. For the Eulerian observers $\hat{n}^a$, we have

$$\omega^{ab} = 0, \tag{2b}$$

$$\theta^{ab} = \sigma^{ab} + \tfrac{1}{3}\theta h^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & -K^{ij} \end{pmatrix}, \tag{2c}$$

$$h^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix}, \tag{2d}$$

$$\zeta^a = (0, \gamma^{ij}\partial_j \ln\alpha), \tag{2e}$$

where K^{ij} is the extrinsic curvature tensor of the time slice. We see that if the time slices are not geodesically parallel ($\alpha = 1$), then *the Eulerian observers are accelerated* and not inertial as in flat spacetime.

Continuing the fluid analogy, one can write down the propagation equations along $\hat{n}^a$ for the above quantities [10]:

$$\mathcal{L}_n h_{ab} = 2\theta_{ab}, \tag{3a}$$

$$\mathcal{L}_n\theta_{ab} = \theta^c_a\theta_{cb} + n^c R_{cabd}n^d + D_a\zeta_b + \zeta_a\zeta_b, \tag{3b}$$

† Latin letters from the first part of the alphabet range over 0,1,2,3; those from the middle part of the alphabet range over 1,2,3. We also set $G = c = 1$.

where $\mathcal{L}_n$ is the Lie derivative along $\hat{n}^a$ and R_{cabd} is the Riemann tensor of g_{ab}. With the slicing (α) fixed, one can then use the identity

$$\mathcal{L}_{\alpha n} = \alpha \mathcal{L}_n = \mathcal{L}_t - \mathcal{L}_\beta, \tag{4}$$

where

$$t^a = \alpha \hat{n}^a + \beta^a = (\alpha^2 - \beta^2)^{-1/2} \hat{t}^a \tag{5}$$

and $\beta^a = (0, \beta^i)$. This, together with using the Einstein field equations $G_{ab} = 8\pi T_{ab}$ to evaluate the four-dimensional R_{abcd} in terms of three-dimensional quantities, yields the standard ADM form of the Einstein field equations [1,2]:

$$\mathcal{L}_t \gamma_{ij} - \mathcal{L}_\beta \gamma_{ij} = -2\alpha K_{ij}, \tag{6a}$$

$$\mathcal{L}_t K_{ij} - \mathcal{L}_\beta K_{ij} = -\alpha \mathcal{K}_{ij} + \alpha \mathcal{R}_{ij} - D_i D_j \alpha - 8\pi\alpha \mathcal{S}_{ij}, \tag{6b}$$

where we have grouped terms to parallel Eq. (3) by defining [2]:

$$\mathcal{K}_{ij} \equiv 2K_{mi}K^m_j - (\text{tr } K)K_{ij} + \tfrac{1}{4}[(\text{tr } K)^2 - K_{mn}K^{mn}]\gamma_{ij}, \tag{6c}$$

$$\mathcal{R}_{ij} \equiv R_{ij} - \tfrac{1}{4} \text{tr}(R)\gamma_{ij}, \tag{6d}$$

$$\mathcal{S}_{ij} \equiv S_{ij} - \tfrac{1}{2} \text{tr}(S)\gamma_{ij}. \tag{6e}$$

Here S_{ij} is the stress of the energy–momentum tensor T_{ab} (see below). It is clear that *the Einstein field Eqs.* (6) *can be regarded as propagation equations for the shear and expansion of the Eulerian observer congruence.* In fact, the expansion $\theta = -\text{tr}(K)$ obeys the trace of the field equations (6):

$$\mathcal{L}_t(\text{tr } K) - \mathcal{L}_\beta(\text{tr } K) = -\Delta\alpha + \tfrac{1}{4}\alpha[\text{tr}(R) + (\text{tr } K)^2 + 3K_{mn}K^{mn} + 16\pi \text{ tr}(S)], \tag{7}$$

where Δ is the three-covariant Laplacian formed from the three-covariant derivative D_i. Since the specification of the lapse function α defines the time slicing, we see that *the expansion of the Eulerian observers congruence is determined by the time slicing coordinate condition.* Finally, it is often useful to use the four constraint equations of general relativity:

$$D^m[K_{mi} - \gamma_{mi} \text{tr}(K)] = 8\pi j_i, \tag{8a}$$

$$\text{tr}(R) + [\text{tr}(K)]^2 - K_{mn}K^{mn} = 16\pi d, \tag{8b}$$

where j_i and d are defined from T_{ab} in Section III.

III. Hydrodynamics and Thermodynamics

We briefly review here the behavior of a perfect fluid, characterized by a rest-mass density ρ, a specific internal energy ε, and an isotropic pressure P. These are defined in the rest frame of the fluid; we may write the energy–momentum–stress tensor as

$$T_{ab} = \rho h \hat{u}_a \hat{u}_b + P g_{ab}, \tag{9}$$

where the quantity

$$h = 1 + \varepsilon + P/\rho \tag{10}$$

is the specific enthalpy. Because of the lack of standardized notation in the literature we provide in Table I a dictionary that compares our notation with that of several standard works. The quantity h defined in Eq. (10) is an essential quantity in relativistic hydrodynamics. Besides generalizing the notion of specific enthalpy, ρh plays the role of the "effective inertial mass" of the fluid [Eq. (19)]. It can also be considered as the injection energy or chemical potential, an interpretation much used by Thorne [11]. Finally, we show that in the comoving gauge, it is h that determines the kinematics.

Since the Einstein equations (6, 8) are tied to the Eulerian observers, we must boost T_{ab} from the $\hat{u}^a$ frame to the $\hat{n}^a$ frame. To the Eulerian observer the perfect fluid appears imperfect [compare (9) above]:

$$T_{ab} = d\hat{n}_a \hat{n}_b + j_a \hat{n}_b + j_b \hat{n}_a + S_{ab}. \tag{11}$$

That is, the fluid appears to have momentum flux and anisotropic shear stresses. By taking the appropriate contraction of (9) with $\hat{n}^a$ and using Eq. (1d), one can easily calculate (d, j_i, s_{ij}):

$$d = \rho h(\Gamma_2)^2 - P, \tag{12a}$$

$$j_i = -\rho h(\Gamma_2)^2 v_i, \tag{12b}$$

$$S_{ij} = P\gamma_{ij} + \rho h(\Gamma_2)^2 v_i v_j, \tag{12c}$$

which agrees, for instance, with Eq. (125.3) in Landau and Lifshitz [12]. These quantities act as the source terms for Einstein's equations (6, 8).

The basic fluid variables (ρ, ε, P) are augmented by the temperature T and the specific entropy S. The first law of thermodynamics relates these quantities:

$$T dS = d\varepsilon + P d(1/\rho). \tag{13}$$

An alternate form that will prove useful is

$$T dS = dh - (1/\rho) dP. \tag{14}$$

TABLE I

Hydrodynamics notation

Quantity Reference	Landau and Lifshitz [12]	Taub [14]	Misner and Sharp [24]	May and White [3]	Schutz [8]	Lichnerowicz [50]	Carter [46]
Rest-mass density							
ρ	nmc^2	ρc^2	n	ρ	ρ_0	r	ρ
Mass-energy density							
$\rho(1+\varepsilon)$	e	$w=\rho(c^2+\varepsilon)$	ε	$\rho(c^2+\varepsilon)$	$\rho=\rho_0(1+\Pi)$	$\rho=r(1+\varepsilon)$	ρ
Enthalpy							
$h=1+\varepsilon+P/\rho$	w/nmc^2	c^2+i	$h=(\varepsilon+P)/n$	$1+\frac{\varepsilon}{c^2}+\frac{P}{\rho c^2}$	$1+\Pi+P/\rho_0$	$f=1+i$	$\mu=\frac{\rho+P}{n}$
Specific entropy							
S	σ/nmc^2	S	s	—	S	S	s
Momentum							
$S_i/(\rho u^t)$	wv_i	$V^\mu=\left(1+\frac{i}{c^2}\right)u^\mu$	—		$V_\alpha=\mu u_\alpha$	$C_\alpha=fu_\alpha$	$\pi_\alpha=\mu u_\alpha$
Constants							
$G=c=1$	G, c	G, c	$G=c=1$	G, c	$G=c=1$	G, c	$c=1$
Signature	+	−	+	−	+	−	+

Much of the literature on relativistic hydrodynamics considers only *isentropic* flow $dS = 0$. Our discussion will not make this restriction, but we shall point out how equations simplify in that case.

One must specify an equation of state that defines one function $(\rho, \varepsilon, P, S)$ in terms of two others, e.g., $P = P(\rho, S)$. Often, discussions of relativistic hydrodynamics assume a *barotropic* equation of state in which the dependent variable is a function of only one quantity, e.g., $P = P(\rho)$. A special example is the isentropic case, $S =$ constant, mentioned above. This restriction is inappropriate to the physical situation of relativistic collapse when shocks occur. (For a comparison of the barotropic and nonbarotropic equations, see Taub [13].)

The conservation laws for energy and for momentum are contained in the four equations:

$$\nabla^a T_{ab} = 0. \tag{15}$$

Just as the form of T_{ab} takes a simple form in the rest frame of the fluid (defined by $\hat{u}^a$), so do these equations. Following Taub's derivation [14] we first contract (15) with $\hat{u}^b$ and then use the first law of thermodynamics (13a) to obtain

$$\nabla_a(\rho \hat{u}^a) + \rho h^{-1} T \hat{u}^a \nabla_a S = 0. \tag{16}$$

When this is combined with the law of baryon conservation

$$\nabla_a(\rho \hat{u}^a) = 0, \tag{17}$$

we derive that the perfect fluid is *locally adiabatic*:

$$\hat{u}^a \nabla_a S = 0. \tag{18}$$

That is, entropy is constant along the flow lines of the fluid. Note that if the fluid had been assumed to be isentropic ($\nabla_a S = 0$), we would not need to assume baryon conservation as an independent law, but rather would have derived it from (15). The other three equations are obtained by projecting (15) into the rest frame of $\hat{u}^a$ using $\tilde{h}_{ab} \equiv g_{ab} + \hat{u}_a \hat{u}_b$:

$$\rho h \, \hat{u}^a \nabla_a \hat{u}_b = \tilde{h}^a_b \nabla_a P. \tag{19}$$

This is just the Euler equation stating that the specific effective mass (ρh) times the fluid's acceleration ($\hat{u}^a \nabla_a \hat{u}_b$) is the pressure gradient, the fluid version of Newton's law of motion.

In summary, the eight unknown fluid quantities (ρ, ε, P, S, $\hat{u}_a$) are determined by Eqs. (17), (13), equation of state (18), (19), and the restriction $\hat{u}^a \hat{u}_a = -1$. We apologize to the reader for repeating well-known material, but it will be useful to have set notation and to have made the distinctions between adiabatic, isentropic, and barotropic.

A final cautionary note: In the real world, fluids shock. Across these discontinuities the entropy of fluid worldlines change, contrary to the law (18). This is because a fluid is dissipative in a shock front and thus imperfect even in its rest frame, violating the form (9) used to derive (18). Thus, one should think of the fluid as distinct regions, in which the above equations hold, separated by shock fronts, across which jump conditions replace (15) and (17). (For detailed discussion of these jump conditions, see Taub [15] and Thorne [16].)

IV. Lagrangian or Comoving Coordinates

One choice of spacetime coordinates is to force the spatial coordinate trajectories to coincide with the matter flow. In fact, until the work of Wilson [17], all relativistic hydrodynamics used this gauge. Taub [13] has provided a comprehensive review of this choice, which we draw from here. Our purpose is twofold. First, we wish to show how the particular calculations in comoving coordinates made by many authors [3,4,22–34] are all special cases of Taub's general comoving equations. Second, we wish to show how Taub's equations are specializations of the mixed Euler–Lagrange system we use. We divide this section into three parts. The general comoving equations, the spherical case, and the nonspherical case.

A. The Comoving Gauge

The requirement that $\hat{t}^a = \hat{u}^a$ to guarantee comoving coordinates is satisfied if (1b) = (1c) or

$$v^i = \beta^i/\alpha, \tag{20a}$$

$$\Gamma_1 = \Gamma_2 \equiv \Gamma = (1 - \beta_k \beta^k/\alpha^2)^{-1/2}, \tag{20b}$$

$$\hat{u}^a = \hat{t}^a = (\alpha^2 - \beta_k \beta^k)^{-1/2}(1, 0), \tag{20c}$$

$$\hat{u}_a = \hat{t}_a = (\alpha^2 - \beta_k \beta^k)^{-1/2}(-\alpha^2 + \beta_k \beta^k, \beta_i). \tag{20d}$$

These are Taub's Eqs. (15.1) and (15.2) [13] in our notation. By substituting them into the Euler equation (19) and using the first law (14), Taub obtains the comoving gauge equation, his (15.4):

$$\partial_t[h\beta_i(\alpha^2 - \beta_k \beta^k)^{-1/2}] + \partial_i[h(\alpha^2 - \beta_k \beta^k)^{1/2}] + T(\alpha^2 - \beta_k \beta^k)^{1/2} \partial_i S = 0 \tag{21}$$

This will be the master equation for unraveling the many special choices of comoving gauge used by other authors. We note that Taub was not the first

to derive Eq. (21). It goes back at least to Tauber and Weinberg [18], see their Eqs. (127) and (128). Tolman [19] states that one can always find a comoving coordinate system, so the idea dates from at least the 1930s.

Assuming the fluid quantities (h, T, S) are known, the three equations (21) are conditions on the gauge variables (α, β_i). One important drawback of the comoving gauge was noted by Taub [14,20]. From (21), one can see that if shocks form, with discontinuities in (h, T, S), these *must* cause discontinuities in (α, β_i). Thus, the comoving coordinate system will not be a smooth one on spacetime. This major defect will be illustrated below.

The great advantage of the comoving gauge is that the equations of hydrodynamics are automatically integrated. Already we have solved for u^a in terms of (α, β_i). Since from (18) and (20c), $u^a \nabla_a S = (\alpha^2 - \beta_k \beta^k)^{-1/2} \partial_t S = 0$, the entropy is constant along each spatial coordinate trajectory:

$$S = S(x^i), \tag{22}$$

except where these trajectories cross shock fronts. The rest-mass density (or number density) is easily found from (17) and (20c) (Taub [13, Eq. 15.3])†:

$$\rho = \frac{(\alpha^2 - \beta_k \beta^k)^{1/2} f(x^i)}{\alpha [\det(\gamma)]^{1/2}}. \tag{23}$$

Thus, ρ is determined by its initial value and the metric. Having used up the conservation equations (15) and (17), we are left with the first law (13a), which reads as a result of (22):

$$\partial_t \varepsilon = -P \, \partial_t (1/\rho), \tag{24}$$

where $P = P(\rho, S)$ completes the solution of the fluid variables.

If α is fixed by some time slicing condition and β_i is determined by (21), this leaves only γ_{ij} to solve for. This can conveniently be done by using Einstein's equations in the ADM form (6, 8) with source terms given by (12). Taub [21] writes down these equations as his (2.5, 2.6, 2.9). Note that some metric functions may be evolved (6) and some solved for by constraint equations (8).

The only remaining question is that of choosing a time function. We have left α free so that it might be chosen in a manner independent of the matter flow, e.g., by maximal slicing (see below). However, Taub [13] has shown that one can choose a time slicing that depends on the flow and in a sense complements the comoving spatial trajectories. We interpret his choice as follows. The matter flow has a part due to radial flow and a part due to circulation. As we show below, the circulation has contributions from both

† Taub's notation differs from ours as follows: his → ours; $k^{\mu\nu} \to \gamma^{ab}$; $g_{ij} \to \gamma_{ij}$; $N \to \alpha$; $p_i \to \beta_i$; $K_{ij} \to -K_{ij}$; $K \to -K$; $e^{\phi} \to (\alpha^2 - \beta_k \beta^k)^{1/2}$; $v^i \to \beta^i/(\alpha^2 - \beta_k \beta^k)$.

entropy gradients and vorticity. Since the radial flow is hypersurface orthogonal, one can bend the time slice so this part of the flow is at rest in the time slice. This determines α. The remainder of the flow (circulation) is made comoving by use of β_i. From Eq. (15.8) in Taub [13], we find that in our language:

$$\alpha = h^{-1}[1 + h^{-2}(C_k + \theta S,_k)(C^k + \theta S,^k)]^{1/2} \tag{25a}$$

and

$$\beta_i = -h^{-2}(C_i + \theta S,_i) \tag{25b}$$

define Taub's special comoving frame. Note that both $C_k(x^i)$ and $S_{,k}(x^i)$ are determined by initial conditions, with C_k specifying any initial vorticity. To see this one can calculate the twist (2a) of $\hat{\omega}^a$ in these coordinates. (See, e.g., Tauber and Weinberg [18], MacCallum and Taub [21], and Pachner [4].)

B. Spherical Spacetimes in Comoving Gauge

Most general relativistic hydrodynamical calculations have been spherical and used comoving coordinates [3,22–34,38,40]. However, these calculations have always derived their equations after the assumption of spherical symmetry. Here we show how they follow from the general comoving equations valid for multidimensional flow.

The first important calculations were those of Tolman [22] and Bondi [23] where the spherical flow was assumed to be pressureless. Since here $p = \varepsilon = S = C_k = 0, h = 1$ and Taub's special comoving frame (25) becomes

$$\alpha = 1, \qquad \beta_i = 0, \tag{26}$$

which, of course, is the geodesic normal coordinates that are natural for dust spacetimes. As is well known, such time slices become singular as the dust free-falls to infinite density {from Eq. (23) $\rho = [\det(\gamma)]^{-1/2}$}. The three-metric has two functions (γ_{rr}, $\gamma_{\theta\theta} = \gamma_{\phi\phi} \sin^{-2}\theta$). One, γ_{rr}, can be integrated by the constraint equation (8b) to give the mass function [Bondi, Eq. (5)], while the other is evolved using (6). These latter equations can be written analogously to Newton's law of motion [Bondi, Eq. (34)].

In the 1960s there was a burst of activity extending the dust calculations to include pressure. The spherical comoving equations were worked out independently by several authors [3,24–28]. These equations were finite differenced and solved on the computer for supernova collapse [3,27–31] and supermassive star collapse [32–34]. We shall not give a detailed comparison of the methods here since they are all equivalent as far as their kinematics are concerned. We particularly study May and White's [3,28] calculations because they are extensively documented.

Misner and Sharp [24] added an isentropic pressure. One now has only $C_k = S_{,k} = 0$, so the Taub gauge (25) becomes

$$\alpha = h^{-1}, \qquad \beta_i = 0, \tag{27}$$

which is Misner and Sharp's equation (1.5). Note that as the collapse proceeds, α becomes less than one in the central region since $\varepsilon \geq 0$ and $P \geq 0$. May and White [3,28] extended this by dropping the restriction that the equation of state be barotropic. As a result, only $C_k = 0$ and Taub's special gauge is

$$\alpha = h^{-1}[1 + h^{-2}T^2 S_{,k} S^k_{,}]^{-1/2}, \tag{28a}$$

$$\beta_i = -h^{-2} T S_{,i}. \tag{28b}$$

This is *not* the gauge used by May and White since they choose $\beta_i = 0$. They apparently did so because of a remark of Tolman's [19] that in spherical comoving coordinates, one can always set $\beta_i = 0$. This is true, but it places severe restrictions on the region of space–time covered by the coordinate system. To obtain their equations one can use the comoving gauge equation (21), which when $\beta_i = 0$, becomes the lapse equation:

$$\partial_i(\alpha h) + T\alpha S_{,i} = 0. \tag{29}$$

With the use of the first law (14) this is Eq. (26) in May and White [28]. Note when $S_{,i} = 0$, this reduces to Misner and Sharp's gauge (27). We examine the consequences of this gauge choice below.

The fluid equations of May and White can be obtained directly from the general comoving equations discussed in Section IV.A. Two are identical: our (22) and (24) are May and White's [28] (27). Our (23) becomes their (22) on noting that in their metric $[\det(\gamma)]^{1/2} = bR^2$, ($\gamma_{\mu\mu} \equiv b^2$, $\gamma_{\theta\theta} \equiv R^2$, μ is the radial coordinate). As in the dust case of Tolman [22] and Bondi [23], the radial metric function is redefined in terms of $m(r, t)$, which is then solved for May and White's Eq. (38) by the Hamiltonian constraint equation (8b). The two-sphere metric function R^2 and its extrinsic curvature $K_{\theta\theta} = U/R$ are evolved by May and White's Eqs. (33) and (34) [28], which are derived from our (6) combined with (8) (see Misner and Sharp [24], Section VII for derivation).

May and White's beautiful pair of papers [3,28] can be interpreted to yield some general lessons about the use of the comoving gauge. There are two separate failures of this gauge. First, when a shock forms, the lapse function α (or a in May and White's notation) becomes discontinuous.† May and

† V. Moncrief (personal communication) has pointed out that a discontinuous lapse does *not* imply that the time slice itself is discontinuous in spacetime, but rather it is only C^0 across the timelike shock front.

White calculate that across the shock $[ah] = 0$ so that the jump in the lapse a just compensates the jump in the enthalpy h. Even when the shock is in a weak gravitational field, the lapse is still discontinuous (see Eq. (82) in May and White [28]). Their numerical calculation of a neutron star collapse and bounce (May and White [3], Fig. 5) clearly shows this. Note the jump in a is *not* a small effect, but of order the values $1 - \alpha_c$ for the central lapse α_c.

The second failure occurs where continued gravitational contraction leads to the formation of a black hole. In this case the time slicing defined by (29) becomes singular shortly after the black hole forms (May and White [3], Figs. 8–10). The question of which time slicing conditions will avoid singularities has recently been extensively investigated [2,35,36]. As shown by Smarr and York [2], the lapse must fall toward zero in the strong field region sufficiently rapidly that the integrated proper time

$$\tau_p = \int_{t_0}^{t} \alpha(x^i, t')dt' \tag{30}$$

is less than the proper time from the hypersurface $t = t_0$ to the singularity. For instance, Smarr and York showed that $\alpha \sim e^{-t/\tau_0}$ for maximal time slicing (see below) so that τ_p is finite and at least has a chance of avoiding the singularity. (For cases where it fails, see Smarr and Eardley [35].) May and White observed that $\alpha \to 0$ in their gauge as well ([3], Fig. 11), but calculated that it did so only as a power law [3]:

$$a \sim t^{-3(\gamma-1)} \tag{31}$$

for equation of state

$$P = (\gamma - 1)\rho\varepsilon. \tag{32}$$

See May and White [3], Eqs. (30) and (31), noting the misprint that left a t out of the first line of their (31). Thus for a typical collapse to a black hole $\gamma \sim \frac{4}{3}$ and $a \sim t^{-1}$. In this case τ_p *diverges* with t as $\ln t$, so we conclude that *May and White's time gauge* (29) *cannot possibly avoid the singularity*. In addition, the time slice first hits the singularity at $\mu \sim 8M_\odot$ for the $21M_\odot$ collapse studied by May and White, enclosing the central region in a "bag of gold" singularity [37]. The time between when the first trapped surface appears on a time slice ($t \sim 0.6644$, Fig. 17) and when $R \to 0$ ($t \sim 0.6645$, Fig. 9) is $\sim 10^{-4}$ sec or $\sim GMc^{-3}$ for a $21M_\odot$ object as expected.

This second failure can be remedied by choosing a different time slicing (α) and using a comoving shift vector determined by (21). Chrzanowski [38] modified May and White's code to do this. If maximal slicing [39] is used to fix α, then the time slicing will go inside any black holes formed, but will usually [35] remain nonsingular for arbitrarily large times after the black

hole is formed. This is particularly important if one wishes to study accretion onto black holes. Of course, if the matter shocks, a discontinuity will develop in β^i and the first failure of comoving coordinates mentioned above will still occur.

We mention in passing that May and White's code can be applied to nonasymptotically flat space-times. Novikov and Polnarev [40] have recently used a similar code to study black hole formation in the early Universe. The two failures mentioned above occur here as well. Another variation on May and White is to include a transport equation. First, Misner and Sharp [25] and Schwartz [29] added a radiative diffusion term. Wilson [30] improved this by using a general relativistic Boltzmann neutrino transport equation [41] to see if neutrinos could blow off the envelope of a collapsing star.

C. *Nonspherical Spacetime in Comoving Gauge*

When one drops spherical symmetry, one must also drop Tolman's dictum that the shift vector β_i can always be made zero. This is because nonspherical flows can exhibit vorticity, which is nonhypersurface orthogonal. That is, there does not exist a global time slice in which the matter is at rest. The idea of using a shift vector $\beta_i = g_{0i}$ such that coordinates comove with the rotating matter was extensively treated by Tauber and Weinberg [18].

Pachner [4] has used the comoving formalism to try to calculate rotating stellar collapse. His series of papers [4] revived the comoving ideas at about the same time Taub was using them for his variational principles. These works seem to have been independent. Pachner [4] restricts the comoving gauge equation (21) to the isentropic case ($\partial_i S = 0$) by his equation $d\varepsilon/d\rho = P\rho^{-2}$. He then uses it to fix his gauge by breaking (21) into two independent pieces, each of which is set equal to zero: the first term of (21) determines β_i and the second term determines $g_{00} = -\alpha^2 + \beta_i \beta^i$. (The third term vanishes because of isentropic flows.) This leads to the isentropic case of Taub's special comoving frame (25). (Pachner's $e^\chi = h$, $A_i = C_i$, and $A_4 = 1$, in our notation.)

Thus, Pachner's gauge can be seen to be the rotating generalization of Misner and Sharp's spherical gauge (27). As such, it will suffer the same two failures of comoving coordinates discussed above (assuming Pachner dropped his isentropic restriction so shocks could be allowed). However, as we shall see below, the nonspherical comoving gauge has in addition a third failure. When shocks form, the resulting entropy gradient drives circulation and thus vorticity. These tend to characteristically form swirls (see Figs. 1–3). If the spatial coordinates attempt to follow this motion (required by the

comoving assumption), they will become badly snarled and perhaps self-intersect. It seems likely that this lack of smoothness in the spacetime coordinate system will be at least as damaging as the discontinuities that will develop at the shocks themselves.

V. Eulerian or Noncomoving Coordinates

We have shown in the previous section that the requirement that the space–time coordinate system exactly follow the matter flow on spacetime leads to three distinct failures of the smoothness of the coordinates. First, shocks cause discontinuities in (α, β_i). Second, time slices chosen by the matter flow (Tolman–Bondi, Misner–Sharp, May–White, Pachner) become singular when black holes form. Third, in nonspherical flow shocks develop swirling matter flow behind them that tangles coordinate lines. In exchange for this, one can easily integrate the fluid variables by (20), (22)–(24).

From a numerical relativity point of view, it would seem to be very important to retain smooth spacetime coordinates in which to describe the gravitational field. This is particularly important when gravitational waves are generated by the mass motion and must propagate outward through the coordinates. Therefore, we have given up the use of the gauge freedom (α, β_i) to simplify the hydrodynamics equations and instead have used it to simplify the description of the gravitational field itself. The price we pay is that we must now integrate the full Eulerian hydrodynamics equations. However, we would have to do that anyway for multidimensional flows to avoid the third failure of the Lagrangian gauge mentioned above. Therefore, we now describe the equations we use and show how they reduce to the comoving ones in the appropriate limit.

A. Eulerian Hydrodynamics

In order to write down the equations of hydrodynamics in the ADM coordinate system defined by (α, β_i), we use the representation (1). It will be useful to define some intermediate variables [17]:

$$D \equiv \rho \hat{u}^t = \rho \Gamma_2 \alpha^{-1}, \tag{33a}$$

$$E \equiv \varepsilon \rho \hat{u}^t = \varepsilon \rho \Gamma_2 \alpha^{-1}, \tag{33b}$$

$$\hat{u}^a = V^a \hat{u}^t, \qquad V^a = [1, \alpha v^i - \beta^i], \tag{33c}$$

$$S^a = \rho h \hat{u}^a \hat{u}^t = \rho h (\Gamma_2 \alpha^{-1})^2 (1, V^i). \tag{33d}$$

We shall treat (D, E, S^i) as our mass density, internal energy density, and momentum density, respectively. The vector V^i will be called the *transport*

velocity. Note that we have factored out

$$\hat{u}^t = \Gamma_2 \alpha^{-1}, \tag{34}$$

which has red-shift factors from both special (Γ_2) and general (α) relativity.

By substituting (33a) and (33c) in (17) we obtain our equation for mass density

$$\partial_t(D\sqrt{-g}) + \partial_i(DV^i\sqrt{-g}) = 0, \tag{35}$$

where $\sqrt{-g} = \alpha[\det(\gamma)]^{1/2}$ is the determinant of the four-metric. Note that the α in D and in $\sqrt{-g} \equiv \alpha[\sqrt{\det(\gamma)}]^{1/2}$ cancel. Thus, it is sometimes more convenient to evolve $\rho\Gamma_2$ than D (particularly if $\alpha \to 0$ in the strong field region). This equation is the familiar mass-continuity equation.

We rewrite (15) in the equivalent form

$$(\sqrt{-g})^{-1/2}\partial_b(T^b_a\sqrt{-g}) - \tfrac{1}{2}(\partial_a g_{cd})T^{cd} = 0, \tag{36}$$

and use our new variables (33) to write (9) as

$$T^a_b = S^a S_b/S^t + P\delta^a_b. \tag{37}$$

Taking the spatial component $a = i$ yields our momentum equation:

$$(\sqrt{-g})^{-1/2}[\partial_t(S_i\sqrt{-g}) + \partial_j(S_i V^j\sqrt{-g})] + \partial_i P - \tfrac{1}{2}(\partial_i g_{cd})S^c S^d/S^t = 0. \tag{38}$$

The last term gives the gravitational acceleration.

Finally, we use (17) in combination with $\hat{u}^a\nabla^b T_{ab} = 0$ to get our energy equation:

$$\partial_t(E\sqrt{-g}) + \partial_i(EV^i\sqrt{-g}) + P[\partial_t(\hat{u}^t\sqrt{-g}) + \partial_i(\hat{u}^t V^i\sqrt{-g})] = 0. \tag{39}$$

The last term is the change in internal energy caused by "PdV" work.

Note that by the choice of variables (33) our general relativistic hydrodynamic equations (35), (38), and (39) look very similar to nonrelativistic hydrodynamics. Special relativity is hidden in the Γ_2 terms in (D, E, S_i, $\hat{u}^t$). General relativity comes in through the space and time variation of $\sqrt{-g}$ and in the gravitational acceleration term in (38). Having evolved the fluid quantities, one can then reconstruct the source terms for Einstein's equations (12) simply by

$$d = \alpha^2 S^t - P, \tag{40a}$$

$$j_i = -\alpha S_i, \tag{40b}$$

$$S_{ij} = S_i S_j/S^t + P\gamma_{ij}. \tag{40c}$$

We then have a complete system of coupled geometrohydrodynamical equations in (6), (8), (35), (38), (39). (For their use in numerical calculations, see [17,43].)

B. *The Comoving Limit*

In general, β_i is chosen to simplify the gravitational field (see Smarr [42] and Wilson [43], for examples). However, we now show how choosing $\beta_i = \alpha v_i$ reduces our Eulerian hydrodynamics equations to the comoving ones discussed above. In the comoving limit (20), we have

$$D \to \rho(\alpha^2 - \beta^2)^{-1/2}, \tag{41a}$$

$$E \to \varepsilon\rho(\alpha^2 - \beta^2)^{-1/2}, \tag{41b}$$

$$V^i \to 0, \tag{41c}$$

$$S^t \to \rho h(\alpha^2 - \beta^2)^{-1}, \qquad S^i \to 0, \tag{41d}$$

$$S_i \to \rho h\beta_i(\alpha^2 - \beta^2)^{-1}. \tag{41e}$$

The primary simplification is that all transport terms (V^i terms) drop out because the spatial coordinates comove with the matter. Using (41a) and (41c) the continuity equation immediately reduces to (23). Besides losing the transport term in (38), the gravitational acceleration has only one term: $-\frac{1}{2}S^t\partial_i g_{tt}$. In the spherical case of comoving time slices with zero-shift vector, e.g., May and White, this term becomes, using (41d),

$$\tfrac{1}{2}S^t\partial_i g_{tt} \to \rho h\partial_i \ln\alpha = \rho h a_i, \tag{42}$$

where a_i is the spatial component of the four-acceleration vector of the (comoving) Eulerian observer. In this special case $S_i \to 0$, the only other term left in (38) is $\partial_i P$, yielding just the spatial components of the force equation (19). For this reason we refer to the last term in (38) as the gravitational acceleration term. In the general case, with $\beta_i \neq 0$, we can use comoving form of (35) and the first law (14), to show that our momentum equation becomes the comoving gauge equation (21). Finally, our energy equation reduces to (24) when one uses the mass continuity equation.

We point out that it is not necessary to use β_i to obtain a comoving system. Because one is using a finite-differenced grid in numerical calculations, one can move the grid points (I, J = constant) relative to the spatial coordinate trajectories (R, Z = constant). This can be represented as a *grid velocity* V^i_g, which can be set independent of β^i. In practice, V^i_g is used to have the grid move in a quasi-comoving sense. That is, it moves in as (nonspherical) collapse continues, so as to leave good spatial resolution of the flow, but it does not try to follow detailed swirling motion. The term V^i_g comes into the hydrodynamical equations by replacing V^i by $V^i - V^i_g$ in the transport terms. This method of computing hydrodynamics is often referred to as *mixed Euler–Lagrange*. (For details on its use, see Wilson [43].) If $V^i_g = V^i$, then the code becomes Lagrangian, while the shift vector β_i is free to set the spatial coordinates [52].

C. *Einstein Equations*

Because we have *not* used up the shift vector freedom by imposing the comoving gauge, we can use it to simplify the gravitational field. This can be done by imposing conditions on the time derivatives of the three-metric functions. One way [2] minimizes the integrated coordinate shear on a time slice, whereas another [43] sets specific components of the three-metric to zero. The motivation in both cases is to separate out as much unnecessary coordinate shear from the true gravitational-wave degrees of freedom. For our discussion here, the important point is that such an approach leads to *elliptic* equations for β^i. Roughly speaking, this means that β^i will be determined by spatial *integrals* over the fluid variables, so that even if they suffer a step discontinuity, β^i will be continuous, unlike the comoving case.

Similarly, the use of maximal time slicing [39] has been shown to lead to smooth lapse functions α in a wide variety of cases [35,38,42–44]. Maximal slicing is obtained by demanding $\partial_t \operatorname{tr}(K) = \operatorname{tr}(K) = 0$, which when used in (7) yields an *elliptic* equation for α. Thus, this approach of freeing (α, β^i) from the detailed behavior of the matter flow corrects the first failure of comoving slices described in Section IV.B. Furthermore, as noted in that section, maximal time slicing usually avoids hitting spacetime singularities [2,35]. This allows one to study the subsequent accretion onto black holes after their formation. As mentioned, Chrzanowski (38) showed one could combine maximal slicing (α) with comoving shift (β^i) to get rid of the second failure of the comoving gauge.

This use of elliptic equations to solve for *nondynamical* or longitudinal parts of the spacetime metric can be extended to the three-metric and extrinsic curvature. Thus, in the isothermal gauge [42,43], one has only two nonzero components of γ_{ij} and three nonzero components of K_{ij} in a nonrotating axisymmetric spacetime. The Hamiltonian constraint (8b) can be converted into an elliptic equation for one of the γ_{ij}s (Wilson [43], Eq. 47) and the momentum constraint (8a) can be used to obtain two of the K_{ij}s by elliptic equations (Wilson [43], Eqs. 28 and 49). This leaves one γ_{ij} and one K_{ij}, which satisfy a wave equation (Wilson [43], Eqs. 50 and 51).

D. *A Numerical Example of Nonspherical Collapse*

We turn finally to the third problem of comoving coordinates that occurs only for nonspherical flows. This problem of tangling of coordinate lines arises because: "After passing through a shock wave, potential flow [isentropic, irrotational flow] of a gas usually becomes rotational flow." This statement is from Landau and Lifshitz's [12] first page on two-dimensional

gas flow. We quote it only to emphasize how ubiquitous this phenomenon is.

To illustrate this effect, we present here some results on the gravitational collapse and bound of a highly distorted star. This calculation was done using the mixed Euler–Lagrange scheme just described and is discussed by Wilson [43]. The equation of state is given by

$$P = \rho\varepsilon \tag{43}$$

appropriate for stiff matter ($\gamma = 2$), such as a neutron star near bounce. The initial condition was the star was axisymmetric and nonrotating with density profile

$$\rho = \rho_0(\sin kx/x), \qquad x = (\sqrt{R^2 + 2Z^2})^{1/2}, \tag{44}$$

appropriate for a $n = 1$ polytrope. The internal energy was 10% of that required to support the star in the Z direction. (We use cylindrical coordinates Z, R, ϕ.)

The star collapsed from rest and bounced at $T \sim 53M$ ($G = C = 1$) when the central α had fallen to $\alpha_c \sim 0.45$ (see Fig. 7 in Wilson [43]). This was therefore a very relativistic bounce. We present here the variation of the density D and the velocity field V^i as functions of (R, Z, T). Figure 1 shows the contours of density and the velocity field at $T \sim 20M$. One can see that the density contours are flattened oblate spheroids and the matter flow is essentially radial. As the center slows down to bounce, the outer layers run into the core forming a strong shock ($T \sim 45M$), which propagates outward. By $T \sim 70M$, Fig. 2, the shock is rapidly accelerating in the decreasing density gradient. Note that the density contours show that the star has bounced along the Z axis and the core is now highly prolate. Most important one sees that the matter flow behind the shock has become rotational with counterrotating vortex rings above and below the equatorial plane ($Z = 0$). Recall the total angular momentum must still be zero since there is no rotation around the Z axis. By $T \sim 130M$, Fig. 3, the shock has reversed almost the entire velocity field. The vortex ring has moved outward, and in its wake has left another vortex ring rotating in the opposite direction. As the prolate core stops and begins to fall inward along the Z axis and outward along the equator, this vortex is reinforced and remains present until the end of the calculation ($T \sim 250M$).

If this calculation has been done with comoving coordinates, the spatial coordinate trajectories would have ended up in a number of swirling vortex rings. As they stretch and twist the metric functions γ_{ij} measuring the proper distance between the coordinate lines would have developed wild distortions, *even though the gravitational field is fairly smooth.* This would severely complicate the extraction of gravitational radiation information as well as

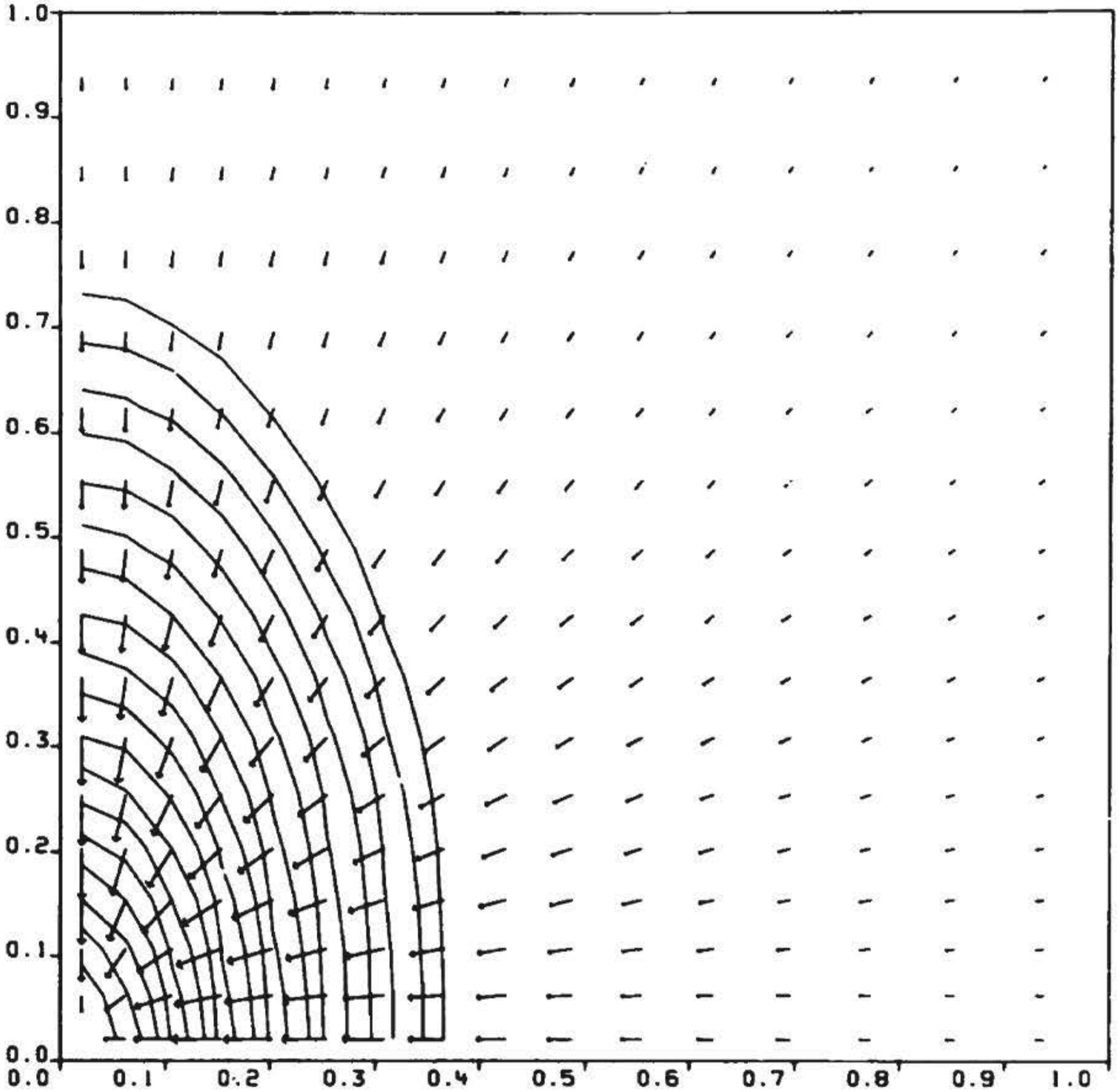


Fig. 1 The collapse of a distorted axisymmetric nonrotating star with a stiff equation of state. The Z axis of symmetry is to the right, while the equatorial plane is vertical. The distances and time scales are in units of $M(G = c = 1)$. This is a snapshot at $T \simeq 20M$ after the collapse from rest started. The density D contours show the initial 2:1 oblate distortion. The arrows represent the velocity vectors V^i of the fluid which at this time is in quasiradial infall (cycle = 50).

distorting its propagation through this grid. By contrast, in the mixed Euler–Lagrange approach, the grid velocity is used to ensure a uniform coverage of the active region. One can see this explicitly since the tails of the velocity vectors are at the grid points. At the same time the shift vector, which would be discontinuous at the shock front in the comoving gauge, is smoothly keeping $\gamma_{RZ} = 0 = \gamma_{RR} - \gamma_{ZZ}$ so that only two metric functions ($\gamma_{RR} = \gamma_{ZZ}$, $\gamma_{\phi\phi}$) need be evolved (instead of the four required in comoving gauge for this problem).

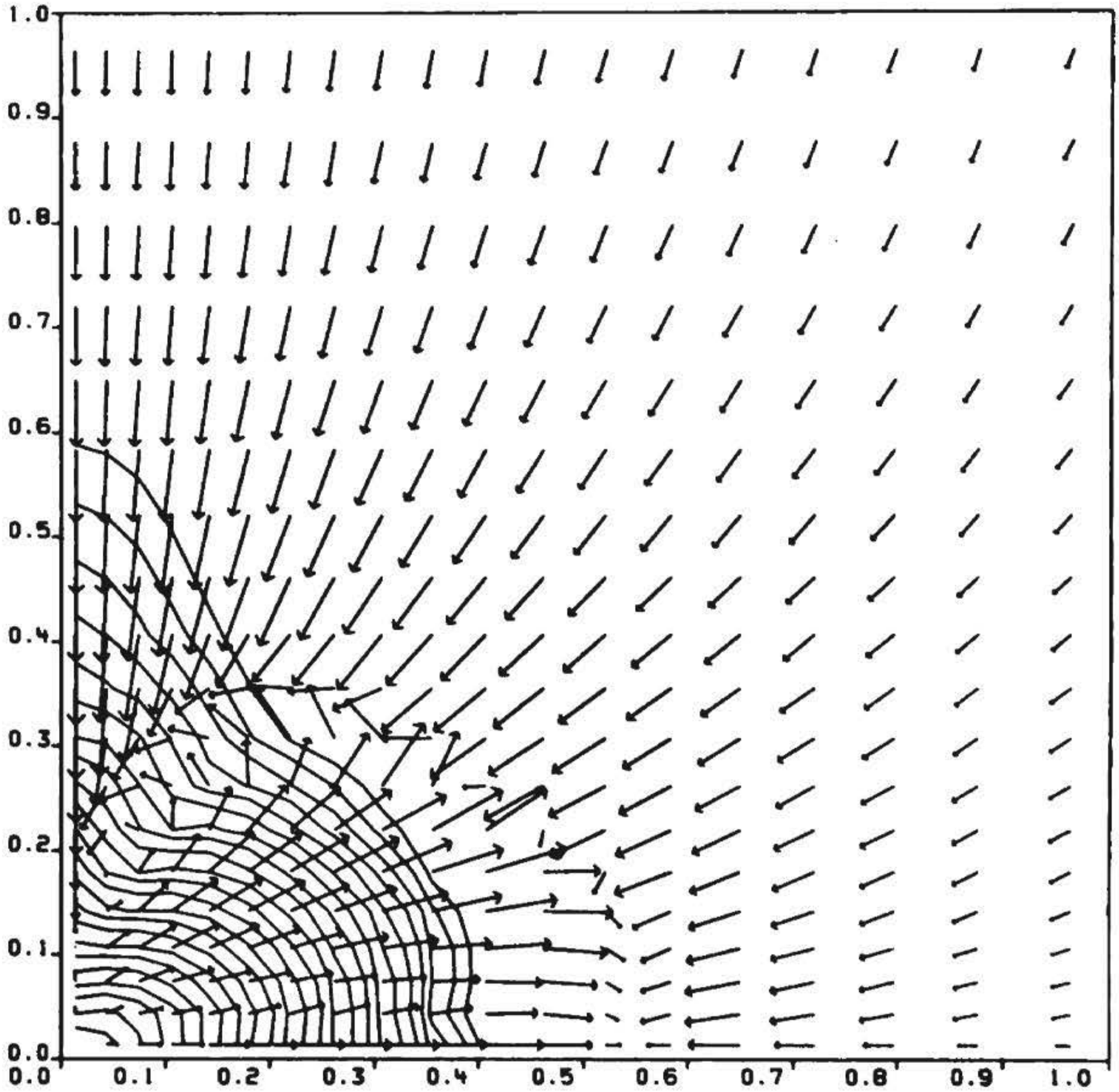


Fig. 2 The same collapse as Fig. 1 now at $T \sim 70M$. The core has bounced in the Z direction and is moving outward causing a shock wave. Behind the shock a swirl has developed due to the entropy gradients produced. The core is now prolate (cycle = 250).

E. Vortices and Supernova Explosion

We shall turn in a moment to a quantitative description of the growth of circulation following a shock, but first we remark on the possible astrophysical significance of such model calculations. Using the most up-to-date physics, one can calculate very detailed *spherical* models of core collapse (see, e.g., Wilson [43]). These include accurate hot equations of state, detailed neutrino physics, and transport. However, the result is that it is not clear whether the shock that forms on bounce can drive off the envelope with certainty. Therefore, it is crucial to determine if deviations from sphericity could introduce new effects.

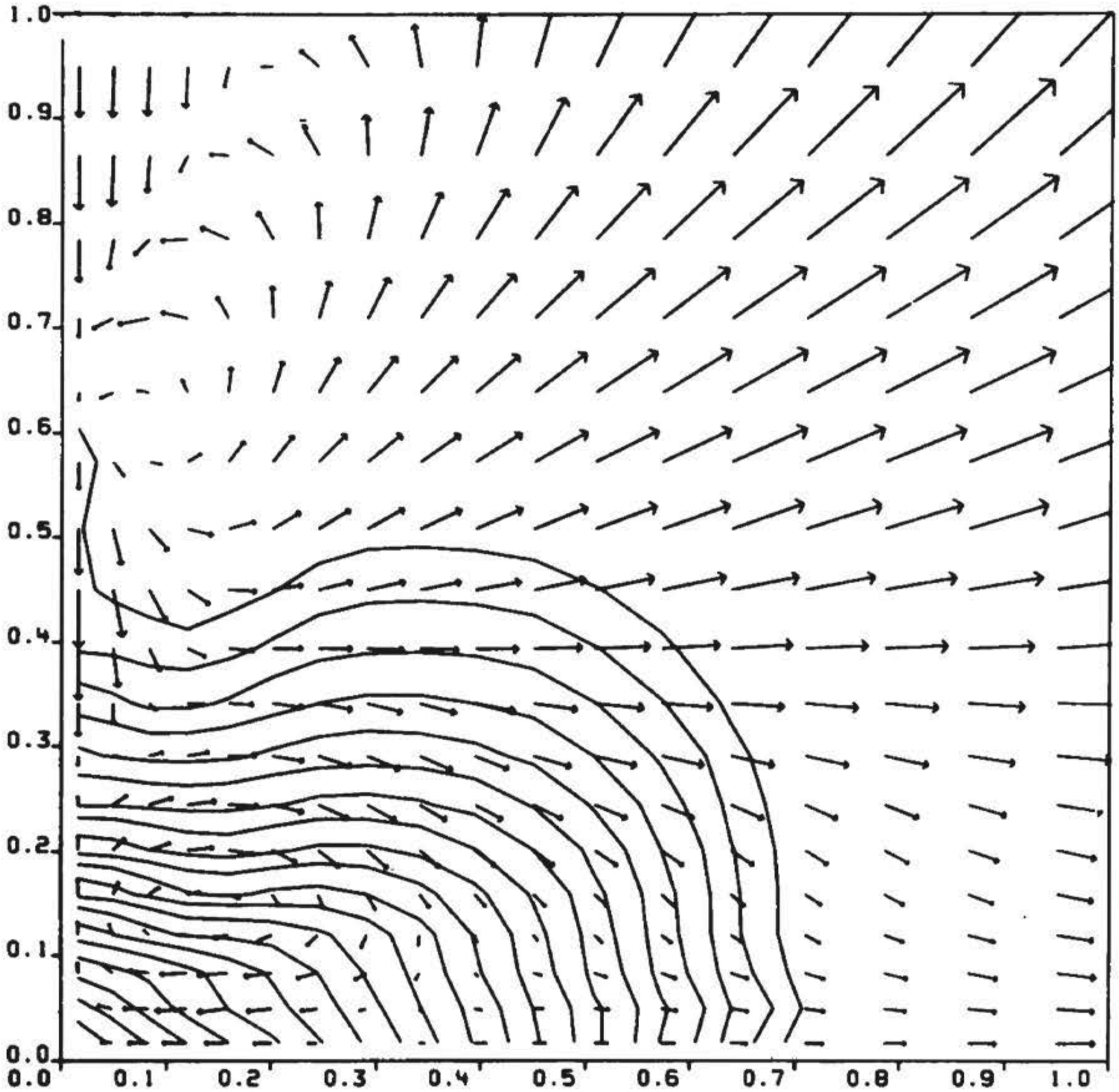


Fig. 3 The collapse is at T ~ 130*M*. The accelerating shock wave has swept off the grid leaving two counterrotating vortices behind it. The core is just about to bounce on the equator (cycle = 450).

Three recently discussed mechanisms may be active in producing a non-spherical configuration after the first bounce in stellar core collapse. First, Saenz and Shapiro [45] have shown that at least under very simple conditions the eccentricity of a bouncing stellar core grows rapidly with each bounce. Second, Epstein [6] has pointed out that after a stellar core has formed, the outside neutronizes rapidly and becomes convectively unstable due to high lepton number on the inside and low lepton number on the outside. If this process overshoots, Colgate [7] has argued that a large-scale Rayleigh–Taylor instability will take place, completely turning over the core by the third bounce. Third, these effects may be enhanced by a peculiar condition

of the core as calculated in the most recent spherical runs [51]. It is found that when matter at densities above $\sim 10^{13}\,\mathrm{g\,cm^3}$ does neutronize, the pressure in the matter may *increase* if the concomitant neutrino losses are not too high. After bounce at $\sim 10^{14}\,\mathrm{g/cm^3}$, the core density falls toward $\sim 10^{13}\,\mathrm{g/cm^3}$. If convection exists and enables a piece of the core to lose some neutrinos, its pressure will rise and it will acquire further bouyancy. Thus, if these three mechanisms can work in concert, it might be possible to reduce large vortical motion in the core.

One of us (JRW) is using a detailed spherical supernova code to see if these particular convection processes work with the restriction to near-spherical symmetry. However, we want to point out here that even in a purely hydrodynamical collapse (no neutrino transport) with the simplest (unrealistic) equation of state, *the nonspherical shock itself drives convection strongly.* Note that two swirls have already occurred (Fig. 3) before the second bounce. It seems likely that this behavior will occur in any nonspherical collapse. Note that in purely spherical collapse the shock front is at right angles to the flow and this induces no circulation. This generic effect may also help Shapiro's mechanism [45] for amplification of core eccentricities during subsequent core bounces. We are currently studying this problem in more detail.

F. The Circulation Theorem

In order to quantify this generation of vortices we briefly review Taub's [5] relativistic generalization of the classical circulation theorem. (For the nonrelativistic case see, e.g., Landau and Lifshitz [12].) Consider a closed contour with path parameter τ and spacelike tangent vector λ^a. Then the circulation around that contour is

$$C = \oint h\hat{u}_a \lambda^a \, d\tau. \tag{45}$$

Note that the integrated quantity is the enthalpy weighted four-velocity. Let the contour be carried into the future along the fluid lines. Then if s is the proper time along the fluid lines, the time rate of change of C is

$$\frac{dC}{ds} = \oint T S_{,a} \lambda^a \, d\tau, \tag{46}$$

that is, entropy gradients can drive the growth of circulation.

Taub defines a convenient antisymmetric tensor:

$$\Omega_{ab} = \nabla_a(h\hat{u}_b) - \nabla_b(h\hat{u}_a) \tag{47a}$$

$$= -2h\omega^d \hat{u}^c \eta_{dcab} + T(\hat{u}_b S_{,a} - \hat{u}_a S_{,b}), \tag{47b}$$

where η_{dcab} is the alternating tensor and ω^d is the vorticity vector:

$$\omega^d = \tfrac{1}{2}\eta^{dcab}\hat{u}_c\hat{u}_{a,b}. \tag{48}$$

By Stokes theorem, C is equivalent to the area integral over Ω_{ab}. Thus, the initial circulation vanishes only if the flow is *both* irrotational and isentropic, i.e., potential flow. Applying Stokes theorem to (46) one can show that the circulation is conserved only if the fluid is barotropic [5].

As Taub [5] describes, this cannot be the case when shocks are present. Even if the flow is isentropic before the shock occurs, a general shock front will leave an entropy gradient behind it, which will drive circulation, just as we saw in Figs. 1–3. Furthermore, if the hydrodynamics is nonspherical, the microphysics will evolve nonspherically, setting up additional entropy gradients. We are studying our collapses to verify that they quantitatively agree with (46).

We note in passing that all of this analysis can be performed very elegantly using differential forms (see Carter [46]). He finds that the quantity $hu_a \equiv \pi_a$ is the natural momentum for a fluid. His circulation theorem (3.9) states that the circulation is *always* conserved *if* one uses as time derivative not proper time $\hat{u}^a\nabla_a$, but rather $1/T\hat{u}^a\nabla_a$.

VI. The Velocity Potential Formalism

Let us summarize the methods for solving hydrodynamics. In the comoving approach (Section IV) the three spatial components of the fluid's four-velocity are set equal to zero. The fourth component is fixed by $\hat{u}^a\hat{u}_a = -1$. The entropy S is constant along each world line and the rest-mass density ρ is obtained from the variations in the spacetime metric from (23). The energy density ε is calculated from the first law (13, 24) with $P = P(\rho, S)$ given by the equation of state. In the Eulerian approach ρ and ε are evolved as D and E in Eqs. (35) and (39). The entropy is then calculated from the first law with $P = P(\rho, \varepsilon)$ as the equation of state. The spatial components of the four-velocity are solved for as S_i the momentum density, with the fourth component again set by the four-velocity normalization.

The velocity potential formalism is a third method, which combines some features of both comoving and Eulerian methods. In this approach one regards the fluid as represented by a set of six scalars on spacetime. These scalars satisfy evolution equations similar to the Eulerian ones. However, the scalars are chosen in such a way that most of them are constant along the fluid worldlines. The formalism goes back to Clebsch [47] as updated by Lin [48] and Seliger and Whitham [49]. Schutz [8] worked out the general relativistic version that we rewrite in our language. See also [53].

The six scalars will be denoted by $(\rho, S, \theta, \phi, \chi, \xi)$. The first two are just the rest-mass density and entropy. We shall assume all other thermodynamic quantities (ε, P, h, T) are functions of (ρ, S) and can be obtained by equation of state or the first law. The other four scalars represent the four components of the four-velocity in the following representation:

$$\hat{u}_a = h^{-1}[\partial_a \phi + \theta \partial_a S + \chi \partial_a \xi]. \tag{49}$$

In the case of isentropic irrotational flow, this reduces to the standard representation of potential flow $V = \nabla\phi$ discussed, e.g., in Landau and Lifshitz [12]. Allowance for entropy gradients is made with (θ, S) and for vorticity by (χ, ξ).

The evolution equations for the scalars (see Eq. (2.22) in Schutz [8]), are rewritten in our Eulerian form by use of (1c) to write the fluid proper time derivative as

$$\hat{u}^a \partial_a = \Gamma_2 \alpha^{-1}(\partial_t + V^i \partial_i). \tag{50}$$

The six equations become

$$\partial_t(D\sqrt{-g}) + \partial_i(DV^i\sqrt{-g}) = 0, \tag{51a}$$

$$\partial_t S + V^i \partial_i S = 0, \tag{51b}$$

$$\partial_t \chi + V^i \partial_i \chi = 0, \tag{51c}$$

$$\partial_t \xi + V^i \partial_i \xi = 0, \tag{51d}$$

$$\partial_t \phi + V^i \partial_i \phi + \alpha \Gamma_2^{-1} h = 0, \tag{51e}$$

$$\partial_t \theta + V^i \partial_i \theta - \alpha \Gamma_2^{-1} T = 0. \tag{51f}$$

To evolve from one time slice to next, one takes the scalars and their spatial derivatives and constructs the spatial components of $\hat{u}_i$. From these and the metric $(\alpha, \beta^i, \gamma_{ij})$, one easily calculates

$$\Gamma_2 = (1 + \hat{u}_i \gamma^{ij} \hat{u}_j)^{1/2}, \tag{52a}$$

$$v^i = \hat{u}_i \Gamma_2^{-1}, \tag{52b}$$

$$V^i = \alpha v^i - \beta^i. \tag{52c}$$

These then allow the calculation of the time derivatives for all six scalars in (51).

In the comoving limit we recover (22) and (23) for S and ρ, plus

$$\chi = \chi(x^i), \tag{53a}$$

$$\xi = \xi(x^i), \tag{53b}$$

$$\partial_t \phi + (\alpha^2 - \beta_k \beta^k)^{1/2} h = 0, \tag{53c}$$

$$\partial_t \theta - (\alpha^2 - \beta_k \beta^k)^{1/2} T = 0, \tag{53d}$$

such that again the hydrodynamics is automatically integrated. The proof that the velocity potential evolution equations are equivalent to the standard ones is given in Schutz [8].

The circulation theorem can be rewritten in this language (see Schutz [8]). The circulation itself is given by

$$C(s) = \oint [\alpha\beta_{,a}\lambda^a + \theta S_{,a}\lambda^a]d\tau,$$

which clearly shows that it is composed of vorticity (α, β) and thermal circulation (θ, S). The time derivative (46) is easily obtained since $dC/ds = \mathfrak{U}^a\partial_a C$ and $\mathfrak{U}^a\partial_a$ on (α, β, S) vanish, while $\mathfrak{U}^a\partial_a\theta = T$.

The velocity potential representation has not been used much for numerical calculations. It is not clear whether it would be more useful than the Eulerian approach. The major area that should be worked out before it can be implemented is how to handle shocks. Both the comoving [28] and the Eulerian [43] schemes use artificial viscosity. The justification for this is given by May and White [28] by first working out the equations for an *imperfect* fluid, which allows for viscosity and heat transport. Such a calculation would be necessary for the velocity potential scheme (presumably adding new scalars), before one would be sure how to add artificial viscosity. We are currently investigating this possibility.

VII. Conclusions

We have attempted to present a unified treatment of the methods researchers have used to study relativistic hydrodynamics. Our emphasis has been on the form of the equations and the kinematic implications of gauge choices because these have important consequences for numerical calculations. The interplay between theoretical formalisms and "numerical experiments" is nicely illustrated by the growth of circulation behind shocks. These results may have interesting astrophysical applications that we shall turn to in a future paper.

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