General Relativistic Magnetohydrodynamics

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ABSTRACT

Jim Wilson pioneered the use of supercomputers to study magnetohydrodynamic (MHD) flows near black holes. In this paper, we analyze the differential equations of general relativistic MHD. We use the "3 + 1" formalism to recover the physical meaning of each term in the equations. Limits are taken to recover the special relativistic Alfvén equation and the static balance of magnetic and pressure forces. Particular emphasis is placed on the role of the anisotropic magnetic inertia. A new form for the equations is found which may simplify future numerical work.

I. INTRODUCTION

Jim Wilson performed a systematic numerical exploration of the physics of accretion onto black holes during the 1970s. These pioneering efforts showed that one could use computer codes to solve the fully nonlinear, time dependent, coupled partial differential equations of general relativistic magnetohydrodynamics. In 1972, Wilson (1972) wrote the first code to handle two-dimensional (2-D) general relativistic (GR) hydrodynamic accretion onto a Kerr black hole. In a companion paper in this volume, Hawley, Smarr, and Wilson explain how, ten years later, that code has been updated and used to investigate fat pressure supported disks around black holes.

In a series of papers, Wilson expanded his technique to allow

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time-varying strong gravitational fields with and without magnetic fields (Wilson 1975, 1977, 1978). These calculations demonstrated that magnetic torques and magnetospheres (Damour et al. 1978, Ruffini and Wilson 1975) could be important around black holes.

This paper is the first in a projected series on the updated version of Wilson's magnetohydrodynamic (MHD) codes. Herein we lay out the basic physics which occurs when magnetic fields are combined with gravitation. The next phase of this refinement will be to find a set of analytic solutions to the fully general relativistic MHD equations. We can then subject the MHD code to the same rigorous updating and calibration procedure which was given the Wilson hydrodynamics code (Hawley et al. 1984a,b).

Thorne and Macdonald (1982; hereafter TM), in an alternative treatment of electromagnetic fields near a black hole, emphasize how difficult it is for an astrophysicist to obtain an intuitive feel for the electromagnetic field variables when they are expressed in terms of the electromagnetic field tensor $\underset{\sim}{F}$ and the 4-current $\vec{J}$. Therefore, as they do, we will express the electromagnetic field in terms of the general relativistic analogs of the familiar electric field $\vec{E}$ and the magnetic field $\vec{B}$ before we look for solutions to the MHD equations. We can then generalize the vast background of work in Newtonian MHD to solve the general relativistic equations in the same way that TM have generalized the powerful integral theorems of electromagnetism to find potentials which generate solutions to Maxwell's equations (Macdonald and Thorne 1982; hereafter MT).

With this as motivation, we do the following. In section II we introduce the 3+1 formalism, which allows us to express our problem in terms of the projection of the four-dimensional fields into a three-dimensional absolute space (see also TM). We then express $\underset{\sim}{F}$ in the 3+1 language and derive relativistic expressions for $\vec{J}$ and $\vec{J} \cdot \underset{\sim}{F}$ (the Lorentz force) in terms of $\vec{E}$ and $\vec{B}$. We also find the electric field in a perfectly conducting fluid in terms of the fluid velocity $\vec{v}$ and $\vec{B}$. In section III we derive the MHD equations, by adding the expression for the Lorentz force derived in section II to the momentum equation, and then find solutions in the static, weak gravitational field limit.

Section IV contains an analysis of and solutions to the static MHD equations, a derivation of and solutions to the equations for Alfvén wave propagation in a uniform background, and a discussion of the dispersion relation for Alfvén waves in a non-uniform medium. In section V we manipulate the momentum equation into a form suitable for finite differencing using an explicit, operator-split code, and show that the displacement current term in the Lorentz force generates an anisotropic inertia in the equations of motion for the matter flow (this due to flux freezing). Finally, in section VI, we discuss some analytic code checks which can be obtained for the solutions above. We compare the "natural" potentials used by Wilson (1975) with those used by MT. Throughout this paper we assume a perfectly conducting fluid moving in a stationary metric, and that G=c=1.

II. THE 3+1 FORMALISM

In this paper, we will use the 3+1 formalism as it is developed elsewhere (TM, York 1979, Smarr et al. 1980, Hawley et al. 1984a). (For a discussion of the history of 3+1, see TM.) This formalism chooses a set of fiducial observers and projects all four-dimensional objects onto and orthogonal to the observers' 4-velocities. Physically these are the quantities which the observers would "see" in their rest frame. We will use the global stationary time slices to define the fiducial observers.

The 4-metric, just like any other second rank tensor, can be projected in this way. This has the effect of splitting the 10 functions in the 4-metric into 1+3+6 functions. These components are called the lapse (α), shift vector ($\vec{\beta}$), and 3-metric (γ). We will use the notation of Smarr et al. (1980) unless otherwise noted. Accordingly, we denote by n^μ (greek indices= t,x,y,z; latin= x,y,z) the unit 4-velocity of the fiducial observers normal to the time slices, with a^μ as the 4-acceleration of the fiducial observers, t^μ as the vector ∂/∂_t in the time direction, and u^μ as the unit 4-velocity of the matter flow. The components of these tensors are tabulated below (Smarr et al. 1980, p. 159):

$$g^{\mu\nu} = \begin{matrix} -1/\alpha^2 & \beta^i/\alpha^2 \\ \beta^j\alpha^2 & \gamma^{ij} - \beta^i\beta^j/\alpha^2 \end{matrix}, \quad g_{\mu\nu} = \begin{matrix} -\alpha^2+\beta_k\beta^k & \beta_i \\ \beta_j & \gamma_{ij} \end{matrix} \qquad (2.1a)$$

$$n^\mu = (1/\alpha,\ \beta^i/\alpha), \qquad n_\mu = (-\alpha,\ \vec{0}) \tag{2.1b}$$

$$a^\mu = (0,\ \gamma^{ik}\partial_k \ln\alpha), \qquad a_\mu = (\beta^k \partial_k \ln\alpha,\ \partial_i \ln\alpha) \tag{2.1c}$$

$$t^\mu = (1,\ \vec{0}), \qquad t_\mu = (-\alpha^2 + \beta_k\beta^k,\ \beta_i) \tag{2.1d}$$

$$u^\mu = W\alpha^{-1}\ (1,\ \alpha v^i - \beta^i), \qquad u_\mu = W(-\alpha + \beta_k v^k,\ v_i) \tag{2.1e}$$

$$\Gamma = -n \cdot \hat{t} = (1 - \beta_i\beta^i/\alpha^2)^{-1/2} \qquad W = -n \cdot u = (1 - v_k v^k)^{-1/2}, \tag{2.1f}$$

where $\hat{t}$ is the unit vector along t.

Note that our W is Smarr *et al.*'s $\Gamma_2 = u^t\alpha$, while our Γ is Γ_1 of Smarr *et al.* Physically, α is the gravitational redshift, W is the special relativistic redshift for matter flows, $\vec{\beta}$ is the velocity at which the coordinate system flows in each time slice, and χ measures distances and angles in the 3-space.

In order to project into time slices normal to n^μ, we introduce the projection operator $h^\mu{}_\nu$ and the three dimensional antisymmetric tensor $\varepsilon^{\mu\nu\gamma}$:

$$h^\mu{}_\nu = g^\mu{}_\nu + n^\mu n_\nu, \tag{2.2}$$

$$\varepsilon^{\mu\nu\gamma} = \varepsilon^{\mu\nu\gamma\delta} n_\delta. \tag{2.3}$$

A spatial 3-tensor is created by dotting all the indices on a 4-tensor with $h^\mu{}_\nu$. The upstairs time component of any spatial 3-tensor is zero, since $n_\mu h^\mu{}_\nu = 0$ and $n_i \equiv 0$ (see Eqn. 2.1b). We can also introduce differential operators D_μ and $\pounds_A$ which generalize the space and time partial derivatives of flat space. These are defined by:

$$D_\mu T^{\mu\nu}{}_\rho = h_\mu{}^\alpha h^\mu{}_\beta h^\nu{}_\delta h_\rho{}^\sigma \nabla_\alpha T^{\beta\delta}{}_\sigma, \tag{2.4}$$

$$\pounds_A T^\mu{}_\nu = A^\delta \nabla_\delta T^\mu{}_\nu - T^\delta{}_\nu \nabla_\delta A^\mu + T^\mu{}_\delta \nabla_\nu A^\delta. \tag{2.5}$$

D_μ is the covariant 3-gradient, and $\pounds_t = \pounds_{\alpha n + \beta}$ is the Lie derivative in the time direction. We use later that $D_\mu \varepsilon^{\nu\delta\beta} = D_\mu h^{\nu\delta} = 0$.

We can now write $F^{\mu\nu}$ in terms of the electric and magnetic fields observed by n^{μ}:

$$F^{\mu\nu} = n^{\nu}E^{\mu} - n^{\mu}E^{\nu} + \varepsilon^{\mu\nu\delta}B_{\delta} \qquad (2.6)$$

Note how we covariantly divide tensors into "time" components (here E^{μ}) sitting next to n^{ν} and "space" components which vanish (here $\varepsilon^{\mu\nu\delta}$) if hit with n^{ν}. Using the Maxwell equation $\nabla_{\mu}F^{\mu\nu} = 4\pi J^{\nu}$, one can obtain the 4-current in terms of E^{μ} and B^{μ}.

$$4\pi J^{\nu} = n^{\nu}(D_{\mu}E^{\mu}) - (1/\alpha)\ [\pounds_{t} - \pounds_{\beta} + \alpha\Theta]\ E^{\nu}$$

$$+ D_{\mu}\varepsilon^{\mu\nu\delta}B_{\delta} + a_{\mu}\varepsilon^{\mu\nu\delta}B_{\delta} \qquad (2.7)$$

where $\Theta \equiv \nabla_{\mu}n^{\mu}$ is the convergence of the fiducial observers (zero in Kerr).

We now investigate the physical origin of each term in the general relativistic Maxwell equation (2.7). The term $D_{\mu}E^{\mu}$ is the spatial gradient of the electric field, which equals 4π times the electric charge density $\rho = -n_{\nu}J^{\nu}$. Thus the time component of equation (2.7) is the "Gauss's law" Maxwell equation. The spatial terms are the "Ampere's law" Maxwell equation. The term $D_{\mu}\varepsilon^{\mu\nu\delta}B_{\delta}$, or curl $\vec{B}$, is the familiar magnetic source of the spatial 3-current. The term in brackets involving Lie derivatives of the electric field is the general relativistic equivalent of the displacement current (equal to the left hand side of Eqn. (3.4c) in TM). The term $\pounds_{t}E^{\mu}$ is equivalent to $\partial_{t}E^{\mu}$; $\pounds_{\beta}E^{\mu}$ is the rate of change of E^{μ} caused by "frame dragging"; and Θ is the time rate of change of the volume element. In flat space, "observers at rest" are also "inertial (non-accelerated) observers." In curved space, however, observers "at rest" (n^{μ}) are in general noninertial (like standing on Earth). Their acceleration (a_{μ}) is necessary to balance the gravitational force trying to pull them from rest. The last term in equation (2.7) is such a noninertial term (see the discussion in section 3.2 of TM for more detail).

Equations (2.6) and (2.7) can now be used to write down the Lorentz force $J^{\nu}F_{\nu\mu}$:

$$4\pi J^{\nu}F_{\nu\mu} = n_{\mu}[-\alpha^{-1}E_{\nu}(\pounds_t - \pounds_\beta + \alpha\theta)\ E^{\nu} + E_{\nu}(D_\delta + a_\delta)\varepsilon^{\delta\nu\beta}B_\beta]$$

$$+ (D_\delta E^\delta)\ E_\mu$$

$$+ B^\delta\varepsilon_{\delta\mu\nu}[\alpha^{-1}(\pounds_t - \pounds_\beta + \alpha\theta)\ E^{\nu} \qquad (2.8)$$

$$- (D_\delta + a_\delta)\ \varepsilon^{\delta\nu\beta}B_\beta].$$

The spatial part is the electric field times the charge density (second line) plus the magnetic field crossed with the spatial current terms (third line) from equation (2.7). If we assume perfect conductivity (no electric field in the rest frame of the fluid), we can use $U_\mu F^{\mu\nu}=0$ to write

$$E^\mu = (1/W)\varepsilon_{\mu\nu\delta}B_\nu U_\delta. \qquad (2.9)$$

This is equivalent to the nonrelativistic equation $\vec{E} = \vec{B} \times \vec{V}$. If we so desired, we could now substitute equation 2.9 into equation 2.8 and obtain an expression for the Lorentz force in a perfectly conducting medium dependent only upon the magnetic field (along with fluid velocity and metric terms).

III. STATIC MHD

Wilson (1976) writes down the general relativistic MHD equations for a perfect fluid with perfect conductivity. We have changed his basic variables by a factor of α as is shown below (see e.g. Hawley et al. 1984a - hereafter HSW):

$$D = W\rho, \qquad (3.1a)$$

$$E = W\varepsilon\rho, \qquad (3.1b)$$

$$S_\mu = W\rho(1 + \varepsilon + P/\rho)\ U_\mu \equiv W\rho h U_\mu. \qquad (3.1c)$$

where ρ is the density, ε is the specific internal energy, and P is the isotropic pressure of the fluid.

The equations of hydrodynamics are discussed in HSW using these definitions, and we shall use their equations (17), (20) and (25) as

our equations of hydrodynamics. In order to obtain the MHD equations, we add the Lorentz force to the momentum equation [HSW's Eqn. (25)]. Because we have a perfectly conducting medium, the energy equation [HSW's Eqn. (20)] remains unchanged (since there is no Joule heating in a medium with zero resistivity). If the resistivity were not zero, then we would have to include a term involving $J^{\mu}U^{\nu}F_{\mu\nu}$ in the energy equation.

We now assume that we are working with a perfectly conducting medium in a Schwarzschild geometry (for details on this geometry see e.g. Misner, Thorne, and Wheeler 1973). This implies that $\Theta = \vec{\beta} = 0$ and that $g_{\mu\nu}$ is diagonal. Next we combine HSW's momentum equation (25) with our equation (2.8) (while writing D_{μ} in terms of ∂_{μ} and using spatial indices when possible) to get

$$\partial_t S_j + \gamma^{-1/2}\partial_i(\gamma^{1/2}S_j v^i) + \alpha\partial_j P - (S^{\mu}S^{\nu}/2S^t)\,\partial_j g_{\mu\nu} + \alpha(D_i E^i/4\pi)E_j$$

$$- (B^i/4\pi)\varepsilon_{ijk}\partial_t E^k + \alpha(B^i/4\pi)(\partial_j B_i - \partial_i B_j)$$

$$+ \alpha(B^i/4\pi)(a_j B_i - a_i B_j) = 0 \qquad (3.2)$$

where $\gamma \equiv \det(\gamma_{ij})$ is the determinant of the Schwarzschild 3-metric.

We will now look for static solutions to the MHD equations ($\partial_t = v^i = 0$, $U^t = 1/\alpha$ and $S^{\mu} = (\rho h/\alpha)(1,\vec{0})$). The condition of perfect conductivity, $\vec{E} = \vec{v} \times \vec{B}$, implies that $\vec{E} = 0$. In this limit, equation (3.2) reduces to

$$\alpha\partial_j P + \rho h a_j + \alpha(B^i/4\pi)(\partial_j B_i - \partial_i B_j) + \alpha(B^i/4\pi)(a_j B_i - a_i B_j) = 0. \qquad (3.3)$$

The terms in this equation are understandable using Newtonian physics (which should be the case, since in the static limit special relativity plays no role). The force exerted by pressure gradients is $\partial_j P$. The term $B^i(\partial_j B_i - \partial_i B_j)/4\pi \sim \vec{B}\times\nabla\times\vec{B}$ is the static Lorentz force (since $\nabla\times\vec{B}$ measures the current present, and $f_{Lorentz} = \vec{J} \times \vec{B}$). The term $\rho h a_j$ is the gravitational force required to keep the fluid "at rest" (since "at rest" does not mean "inertial"). If we now go to flat space ($\alpha \to 1, a^{\mu} \to 0$), the gravitational acceleration goes away, just as we would expect, and equation (3.3) becomes

$$\partial_j P + (B^i/4\pi)(\partial_j B_i - \partial_i B_j) = 0. \qquad (3.4)$$

The last term is the reason that magnetic field lines are sometimes thought of as strings. Equation (3.4) can be written in the form

$$\partial_i (T^i{}_{j\,\mathrm{Hydro}} + T^i{}_{j\,\mathrm{Mag}}) = 0, \qquad (3.5)$$

where

$$T^i{}_{j\,\mathrm{Mag}} = (1/4\pi)(B^2\delta^i{}_j - B^i B_j) - (B^2/8\pi)\delta^i{}_j. \qquad (3.6)$$

This is the stress tensor for a tension of $B^2/4\pi$ superposed on an isotropic pressure of $B^2/4\pi$ (see e.g. Blandford 1983).

There are two distinct symmetries for which solutions to equation (3.4) are readily found: translational and rotational. If we take all of our fields to be functions of y and z (r and z) in cartesian (cylindrical) coordinates and take the magnetic field to be pointing in the x (ϕ) direction, then $B^i\partial_i$ vanishes and equation (3.4) reduces to

$$\partial_j (P + B^2/8\pi) = 0.$$

Equation (3.7) has the obvious set of solutions $B^2 = 8\pi(P_{tot} - P)$, where P_{tot} is an arbitrary constant which is greater than or equal to the maximum value of P. This solution just balances the force due to magnetic pressure with that due to hydrodynamic pressure, which is what one would physically assume for flat space.

Next we look at MHD in a weak gravitational field with Newtonian potential Φ. The line element for such a spacetime is [Misner, Thorne and Wheeler 1973, Eqn. (16.2a)]

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)dr^2 + r^2 d\Omega, \quad \Phi \ll 1. \qquad (3.8)$$

This means that $\alpha \simeq 1 + \Phi$ and $a_i = \partial_i \ln\alpha \simeq \partial_i \Phi$ (i.e., the gravitational acceleration is the gradient of the gravitational potential). Equation (3.3) becomes

$$\partial_j P + (B^i/4\pi)(\partial_j B_i - \partial_i B_j) + [\rho h \gamma^i{}_j + \{B^2\gamma^i{}_j - B^i B_j\}/4\pi]\partial_i \Phi = 0. \quad (3.9)$$

Notice the term in braces {}. It is from the gravitational field pulling on the energy contained in the magnetic field. Along with the term involving ρh, this term is the anisotropic moment of inertia tensor for the system (see, e.g. exercise 5.4 in Misner, et al. 1973). Even for magnetic fields with the special symmetries described above, it is not easy to get an analytic solution. Even if we assume weak magnetic fields $B^2 << \rho$ and nonrelativistic temperatures $h \approx 1$, the term $\rho\partial_i\Phi$ is not a perfect differential except in the trivial case of constant density everywhere. However, it would be straightforward to solve equation (3.9) numerically.

IV. ALFVÉN WAVES

We wish to obtain some dynamical solutions to the MHD equations. The simplest nontrivial solution is for Alfvén waves without dispersion. With this in mind we will derive the flat space, special relativistic, Alfvén wave equation from our general relativistic formalism. For the purposes of the calculation we will assume that the lapse function α and shift vector $\vec{B}$ vanishes.

Start with an unperturbed system at rest ($V^i \equiv 0$). Let all gradients be zero. If any of these were nonzero, they would generate dispersion (scattering) terms. Note that since we have a perfect conductor and $V^i = 0$, $E^i = 0$. Now introduce a perturbation such that $B^i \to B^i + b^i$, $V^i \to 0 + v^i$, $P \to P + p$, $\rho \to \rho + d$, and $E^i \to 0 + e^i$, and solve the equations of motion for v^i.

The momentum equation (3.2) becomes, to first order in the perturbation,

$$\rho h \partial_t v_j + \alpha\partial_j p - B^k \varepsilon_{ijk}\partial_t e^i + \alpha(B^i/4\pi)(\partial_j b_i - \partial_i b_j) = 0 \quad (4.1)$$

or

$$\partial_t v_j + (1/\rho h)(\partial P/\partial\rho)\partial_j d - B^i\varepsilon_{ijk}(\partial_t v_m)\varepsilon^{mkn}B_n/(4\pi\rho h) +$$

$$B^i(\partial_j b_i - \partial_i b_j)/(4\pi\rho h) = 0, \quad (4.2)$$

where we have substituted equation (2.9) into equation (4.1) and divided by ρh to get equation (4.2). Notice that we include rela-

tivistic temperatures h ≠ 1, but only mildly relativistic velocities since we drop the $\sigma(v^2)$ term which makes $W \cong 1$. We also need the equation of continuity to eliminate $\partial_j d$:

$$\partial_t d + \rho \partial_i v^i = 0 \tag{4.3a}$$

$$\partial_t \partial_j d + \rho \partial_j \partial_i v^i = 0. \tag{4.3b}$$

Taking ∂_t of equation (4.2) yields:

$$\partial_t{}^2 v_j + (c^2/\rho h)\partial_t \partial_j d + B^i \varepsilon_{ijk} B_m \varepsilon^{mkn} (\partial_t{}^2 v_n)/(4\pi\rho h)$$

$$+ B^i (\partial_j \partial_t b_i - \partial_i \partial_t b_j)/(4\pi p h) = 0 \tag{4.4}$$

where $c \equiv (\partial P/\partial \rho)^{1/2}$. We now need an expression for $\partial_t b_j$. This is obtained by using the other Maxwell equation $\nabla_\mu {}^*F^\mu{}_\nu = 0$, where

$${}^*F^\mu{}_\nu = n^\mu B_\nu - n_\nu B^\mu + \varepsilon^\mu{}_{\nu\delta} E^\delta. \tag{4.5}$$

This yields (with $\vec{\beta} = \Theta = \sigma_{\mu\nu} = \omega_{\mu\nu} = a_\mu = 0$ appropriate for flat spacetime)

$$\partial_t b_i = D_j \varepsilon_i{}^{jk} e_k. \tag{4.6}$$

Equation 4.6 is the special relativistic form of the Maxwell equation $\partial_t \vec{b} = D \times \vec{e}$. Substituting equations (4.3b) and (4.6) into equation (4.4) and using equation (2.9) and $\varepsilon_{ijk}\varepsilon^{mkn} = \delta_i{}^n \delta_j{}^m - \delta_i{}^m \delta_j{}^n$, we get:

$$\{\delta^i_j + [(B^2 \delta_j{}^i - B_j B^i)/(4\pi\rho h)]\}\partial_t{}^2 v_i - (c^2/h)\,\partial_j \partial_i v^i$$

$$+ (B^i \varepsilon_{ijk} \partial_m \varepsilon^{kmn} \partial^p \varepsilon_{npq} B_r \varepsilon^{qrs} v_s)/(4\pi\rho h) = 0. \tag{4.7}$$

This is the special relativistic Alfvén wave equation for mildly relativistic velocities $W \simeq 1$. For a conventional comparison see equation (3-394) of Lang (1980). Again we see the same anisotropic inertia tensor as in equation (3.9), here multiplying $\partial_t^2 v_i$. We can now Fourier transform by letting $\vec{v} = \vec{\varepsilon}\, e^{-i(k\cdot x - \omega t)}$ and obtain

$$\omega^2 \{\hat{\varepsilon} + [V_A{}^2 \hat{\varepsilon} - (\vec{V}_A \cdot \hat{\varepsilon}) \vec{V}_A]\} - (c^2/h)(\vec{k} \cdot \hat{\varepsilon}) \vec{k}$$

$$- \vec{V}_A \times \vec{k} \times \vec{k} \times \vec{V}_A \times \vec{\varepsilon} = 0 \qquad (4.8)$$

where $\vec{V}_A \equiv \vec{B}/(\sqrt{4\pi\rho h}\ W)$ is the special relativistic Alfvén velocity (here $W = 1$).

This equation admits three different classes of solution. The first is where $\hat{k}$, $\hat{\varepsilon}$, and $\hat{V}_A$ are all parallel. Here the cross product term and the magnetic part of the inertia term vanish. This gives longitudinal waves propagating in the direction of $\hat{V}_A$ with a speed $C_{Hydro} = (c^2/h)^{1/2}$. This is the sound speed of a fluid at relativistic temperatures. The factor of h comes from the inertia of the fluid's internal energy.

A second mode is with $\hat{k}$ parallel to $\hat{\varepsilon}$ and perpendicular to $\hat{V}_A$. This corresponds to longitudinal waves propagating perpendicular (along $\hat{k}$) to the magnetic field with a speed of $C_{Magnetosonic} = ([(c^2/h)+V_A{}^2]/(1+V_A{}^2))^{1/2}$. The $V_A{}^2$ term in the numerator is due to magnetic pressure, and the factor of $1+V_A{}^2$ in the denominator is due to the inertia contained in the energy density of the magnetic field.

The final mode has $\hat{k}$ parallel to $\hat{V}_A$ and perpendicular to $\hat{\varepsilon}$ ("Alfvén waves"). This is a transverse mode propagating in the direction of $\vec{B}$ with a speed of $V_{Transverse} = (V_A{}^2/(1+V_A{}^2))^{1/2}$. This speed is a factor of $(1+V_A{}^2)^{-1/2}$ slower than the Alfvén speed obtained in most derivations. This is because most derivations throw away the term in the Lorentz force which arises from the displacement current. We only need to keep that term when V_A is large ($\simeq 1$). It should be noted, however, that it is this term which will prevent the superluminal transfer of information as $V_A \to \infty$.

In the derivation above, we were limited to special relativity by the absence of gradients in our initial (unperturbed) system. If gradients do exist, then they will tend to scatter waves between momentum states. This makes it very difficult to find analytic solutions to even the linearized MHD equations, but we should still be able to predict the propagation of perturbations by using numerical methods. In particular, if the gradients in the background are small, one can find a Green's function for equation (4.7) and use the gradients as forcing terms.

V. THE ANISOTROPIC INERTIA

As we saw above, the term in the momentum equation containing the displacement current will be important in regimes where the Alfvén velocity is of order one. This means that in our numerical simulations of accretion disks, there will be regions which will be field dominated in which we cannot throw out the displacement current term, as is often done in numerical treatments of (GR) MHD. This presents a problem, however, to those using a time-explicit, operator-split scheme. Since the electric field is the cross product of the velocity of the fluid and the magnetic field, the displacement current will contain a term which is dependent on the time derivative of the momentum (through the velocity term). In the old Wilson code, this term was a major problem. It was dealt with by adding the magnetic-electric density of the field components perpendicular to the acceleration to the matter density in the acceleration equation.

With this in mind, we rewrite equation (3.2) as

$$\partial_t S_j = L_j - M_j{}^i \partial_t S_i , \tag{5.1}$$

where L_j contains all of the terms in equation (3.2) which are not proportional to $\partial_t S_i$ and $M_j{}^i$ is the matrix of coefficients of $\partial_t S_i$. In terms of our MHD field variables,

$$M_j{}^i \equiv (B^2 \delta_j{}^i - B_j B^i)/(4\pi\rho h W^2) = (V_A{}^2 \delta_j{}^i - V_{A_j} V_A{}^i), \tag{5.2a}$$

where again $V_A = B/(4\pi\rho h W^2)^{1/2}$ is the Alfvén velocity. We note that $\underset{\sim}{M}$ is just the anisotropic inertia tensor for the magnetic field which showed up in equations (3.9) and (4.7) renormalized by the factor $(4\pi\rho h W^2)^{-1}$. The vector $\vec{L}$ is given by

$$\begin{aligned} L_j \equiv -\Big[& \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma}\, S_j v^i) + \alpha \partial_j P + (S^\mu S^\nu / 2S^t)\, \partial_j g_{\mu\nu} + \alpha (D_i E^i / 4\pi) E_j + \\ & \alpha (B^i/4\pi)(\partial_j B_i - \partial_i B_j) + \alpha (B^i/4\pi)(a_j B_i - a_i B_j) - \\ & (B^k/4\pi)\big[\partial_t (B_k/\rho h W^2)\delta_j{}^i - \partial_t (B_j/\rho h W^2)\delta_k{}^i\big] S_i \quad . \end{aligned} \tag{5.2b}$$

The formal solution to our problem (equation 5.1) is:

$$\partial_t \vec{S} = (\underset{\sim}{I} - \underset{\sim}{M})^{-1} \vec{L} \ . \qquad (5.3)$$

Note that although we are still specialized to Schwarzschild, the following formal analysis should hold for any stationary spacetime. The shift vector will bury itself in the definitions of $\vec{L}$ and $\vec{V}_A$.

Looking at equation (5.2a), we see that $\underset{\sim}{M}$ is the transverse projector on $\vec{V}_A$ and has three eigenvectors. The first eigenvector is parallel to $\hat{V}_A$ and has eigenvalue zero. The other two are degenerate, perpendicular to $\hat{V}_A$, and have eigenvalue $V_A{}^2$. Physically $M_j{}^i$ is the magnetic field's renormalized anisotropic inertia tensor. The field is not affected by fluid motion parallel to it, but it is dragged along by any transverse fluid motion (because of perfect conductivity) and contributes an inertial density of $B^2/4\pi$. Using our eigenvalue analysis, we look for $(\underset{\sim}{I} - \underset{\sim}{M})^{-1}$ of the form

$$(\underset{\sim}{I} - \underset{\sim}{M})^{-1} = [(1 + V_A{}^2)\ \underset{\sim}{I} - \vec{V}_A\vec{V}_A]^{-1} = \kappa I + \zeta \vec{V}_A\vec{V}_A \qquad (5.4)$$

where $\vec{V}_A\vec{V}_A$ is the external product. Solving for κ and ζ, we find:

$$[\underset{\sim}{I} - \underset{\sim}{M}]^{-1} = (\underset{\sim}{I} + \vec{V}_A\vec{V}_A)/(1+V_A{}^2) \ . \qquad (5.5)$$

This matrix has eigenvalues 1 for motion parallel to $\vec{V}_A$ and $1/(1+V_A{}^2)$ for motion perpendicular to $\vec{V}_A$ and can be interpreted as the ratio of fluid inertia to total inertia. Thus we have shown that the anisotropic inertia of the magnetic field enters naturally into the momentum equation.

Using equations (5.5) and (5.2b), we obtain the new form of the momentum equation:

$$\partial_t S_j = [(\delta_j{}^i + V_{Aj}V_A{}^i)/(1 + V_A{}^2)]\ L_i \qquad (5.6)$$

where L_i is defined in (5.2b). In order to use operator splitting on this, we would difference each term in L_i as we would treat the corresponding term in (3.2), then multiply it by $(\delta_j{}^i + V_{A_j}V_A{}^i)/(1 + V_A{}^2)$ before adding it to $\partial_t S_j$. This new method (5.6) is currently being tested numerically to see if it is accurate and stable.

VI. CODE CHECKS

The purpose of the calculations in this paper is twofold. First, we wish to manipulate the MHD equations analytically into the form most suited for finite differencing (i.e., section V). Second, we need analytic solutions to the MHD equations to use as a testbed in all of the different regimes which the code will investigate. Usually, however, the fundamental variables used in analytic and numerical techniques differ, making it messy to go from one formalism to another. The numerical variables can sometimes obscure the physics contained in the difference equations. For example, Wilson's fundamental electromagnetic variables are A_ϕ, where $\hat{A}$ is the 4-vector potential, and $H_\phi \equiv F_{RZ}$ (Wilson 1975). If we write the components of $\underset{\sim}{M}$ using these variables, things become very messy and the underlying symmetries are obscured.

On the other hand, when $\underset{\sim}{M}$ is written in terms of $\hat{B}$, the symmetries become quite clear. In the same way, Macdonald and Thorne's metric components have deep physical significance, although they may not be optimum for coding. To aid in translation between the two formalisms, Table 1 provides a list of equivalent terms in our "numerical" variables and MT's "analytic" variables. Note that Wilson's A_ϕ and H_ϕ are equivalent to Ψ and I (MT's potentials which generate static solutions to Maxwell's equations). This is an example of Wilson's deep intuition in picking the pertinent variables in a problem.

The simplest analytic tests check the ability of a code to keep a static solution static. Most of these will involve balancing magnetic pressure with hydrodynamic pressure or gravitational force according to equation (3.9). One possibility is to take a uniform gas, put a constant gravitational force on it in the z direction, then use the magnetic pressure of a field in the x direction to balance this force. A second is to put a varying pressure on a gas, then balance it with magnetic pressure in the absence of gravitational fields. The gas should not pick up a velocity in either case.

Table 1

EQUIVALENT TERMS: WILSON (1975) VS. MACDONALD-THORNE (1982)
(for Kerr spacetime)

HSW	MT
α	α
β	$-\omega$■
n	u
t	k
$F^{\mu\nu}$	$-F^{\mu\nu}$
g^{tt}	$-1/\alpha^2$
$g^{t\phi}$	$-\omega/\alpha^2$
$g^{\phi\phi}$	$(1-\omega^2\ \tilde{\omega}^2/\alpha^2)/\tilde{\omega}^2$
$g_{\phi\phi}$	$\tilde{\omega}^2$
A_ϕ	* $\Psi/2\pi$
H_ϕ	* $\sqrt{g_{rr}g_{\theta\theta}}\ 2I/\alpha$
$(\alpha/\sqrt{-g})\ \partial_\theta A_\phi$	B^r
$(-\alpha/\sqrt{-g})\partial_r A_\phi$	B^θ
$(\alpha/\sqrt{-g})H_\phi$	B^ϕ

*Defined by MT only for time independent electromagnetic systems.

A scenario which we have not treated is the shock tube (see e.g. HSW). A shock tube with a constant magnetic field perpendicular to the shock front should have the same solution as the shock tube run without the field present. It should also be possible to derive the relativistic jump conditions for a magnetic field parallel to the shock. Another possible test is to substitute magnetic pressure for some of the hydrodynamic pressure in an analytic fat disk, and ascertain that the disk does remain stationary. In this test, however, we must watch out for Rayleigh-Taylor instabilities.

Finally, we can run tests dealing with the propagation of Alfvén waves. The easiest is to check the three modes of propagation in a uniform background. It should also be possible to check the disper-

sion relation obtained in equation 4.13 in simple cases. Either of the static tests outlined above should lend itself easily to such a test as the background. Green's functions, if found for equation (4.7), could be used to provide a critical test on the code's ability to propagate all three kinds of modes.

VII. CONCLUSIONS

We have used the 3+1 decomposition to analyze the MHD methods of Wilson. We find that using this notation clarifies the physical content in a way which neatly parallels that of Thorne and Macdonald (1982). Our emphasis is on the Lorentz force addition to the momentum equation. We find that the magnetic field contributes anisotropically to the inertia term in a way which is physically easy to see. Various limits of the MHD momentum equation are derived (static, Alfvén waves, etc.), and in each case we can see how relativity modifies the well-known low velocity equations. Finally, we present a new "trick" for rewriting the momentum equation in a way we hope will be useful numerically. It is clear that because of the many new degrees of freedom in MHD, the code testing will be a much more involved task than is the case for hydrodynamics alone. On the other hand, the great variety of physical effects which are present in general relativistic MHD were almost all appreciated by Wilson in his original work of almost a decade ago.

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