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Our Elastic Spacetime: Black Holes and Gravitational Waves

A new computer program shows that the old analogy of spacetime as a rubber sheet is remarkably valid

Most people with a scientific education have at some point encountered the statement "Spacetime is curved in Einstein's general theory of relativity." This sounds very mysterious. What is curved spacetime? Is space curved, or time, or both? Curved space is something that many people can imagine (because the standard pictures of the 2-dimensional surfaces of spheres and saddles have been so widely reproduced). But what does curved time mean?

To reach answers to these questions, let us start at the very beginning and review briefly the history of gravitation theories. A convenient starting point is Newton's theory of gravity, the familiar inverse-square force between two bodies of masses m_1 and m_2,

$$\underset{\sim}{F}_1 = \frac{Gm_1m_2}{r^2}\underset{\sim}{\hat{r}}_{12} \qquad (1)$$

This is an action-at-a-distance theory, in which the force exerted on one object by another is transmitted instantaneously, so that the unit vector $\underset{\sim}{\hat{r}}_{12}$ in eq. 1 always points to the current position of one mass relative to the other. An equivalent way of writing the Newtonian force law is to introduce the gravitational potential ϕ in the familiar equations

$$\underset{\sim}{F}_1 = -m_1\nabla\phi$$
$$\phi = Gm_2/r \qquad (2)$$

The potential ϕ is here just a bookkeeping convenience for keeping track of the gravitational forces acting on the mass m_1.

In the latter part of the nineteenth century, however, there was a subtle shift in the *interpretation* of the potential ϕ. The impetus for this shift was the development of electromagnetism from Coulomb's force law (the electrostatic analog of eq. 1) to the electric and magnetic "field" picture of Faraday and Oersted. This new view was finally codified in Maxwell's equations, which invest the electromagnetic fields with a set of dynamical properties all their own—allowing propagating electromagnetic waves (like light). In gravity, this shift in emphasis is reflected by rewriting the second half of eq. 2 in the equivalent form

$$\nabla^2\phi = 4\pi G\rho \qquad (3)$$

where ρ is the density of mass. Eq. 3 is now a *field* equation for ϕ, although one that still embodies the notion of instantaneous action at a distance. Disturbances in the field propagate instantaneously because there are no time derivatives in the equation. The field ϕ has no *dynamical* freedom; it is strictly coupled to its sources ρ. There is a very natural way to give ϕ some dynamics, allowing gravitational waves on much the same footing as the electromagnetic waves of Maxwell's equations. This is to replace eq. 3 by

$$\nabla^2\phi - \frac{1}{v_g{}^2}\frac{\partial^2}{\partial t^2}\phi = 4\pi G\rho \qquad (4)$$

which is a wave equation in which the "velocity of gravity" is v_g.

It may be largely a historical accident that an equation like eq. 4 never received much attention as a theory of gravity. The accident was that Einstein so early recognized the inconsistency between Maxwell's equations and the Galilean-Newtonian notions of absolute time, and he developed his special relativity theory (SRT) to replace these notions. Eq. 4 is inconsistent with SRT because it does not Lorentz-transform correctly: i.e. if one observer finds eq. 4 to be true, then an observer moving at some different velocity will find it to be false—evidently a contradiction.

The inconsistency has two causes: first, the left-hand side is itself not Lorentz invariant. This can be fixed by taking $v_g = c$, postulating that the speed of gravity and the speed of light are the same (conceptually neat!). Second, the left-hand side is a scalar (as opposed to a vector or a tensor) because ϕ is a scalar; but the right-hand side's ρ is not a scalar in SRT. Rather, it is the name given to one particular component of a *tensor*—the matter stress-energy-momentum tensor $T_{\mu\nu}$. One cannot change the fact that matter variables come in the form of tensors. All one can do is invent a left-hand side that is a tensor too. One natural way turns out to be

$$\nabla^2 h_{\mu\nu} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}h_{\mu\nu} = 16\pi G\overline{T}_{\mu\nu} \qquad (5)$$

(A scalar has a single numerical value at every point in space. A tensor has

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16 values, called components, arranged in a 4 × 4 matrix. The indices μ and ν refer to the rows and columns of the matrix.) Here the tensor $h_{\mu\nu}$ has taken over the role of our previous scalar gravitational field ϕ, and it satisfies a wavelike equation. The tensor $\overline{T}_{\mu\nu}$ is closely related to the stress-energy-momentum tensor $T_{\mu\nu}$.

Eq. 5 and its mathematical cousins were actually considered by Einstein and others in the years 1900–1920. The equation is not quite consistent with experiment as it stands above. It predicts a perihelion advance of the planet Mercury that is too large by a factor 4/3, for example. In recent years (with the benefit of hindsight) elementary particle theorists have gone back to eq. 5 and shown that with minor if sophisticated changes, which are uniquely demanded by some internal consistency arguments, the equation becomes consistent with all experiments to date. If this had been the historical development, then the idea of curved space might never have entered gravitation theory. No experimental or observational evidence would have forced us to notice the omission of this concept until the present, the 1970s. Now, the evidence of X-ray astronomy first suggests that some binary star systems can only be understood as a black hole orbiting an ordinary star. Black holes, as we will see, epitomize the concept of curved space in gravitation theory.

That curved space *did* come into the theory, then, seems to be another historical accident. The accident is, first, that Riemannian geometry had been developed and disseminated just within the previous few decades and, second, that Einstein's style of physics was so influenced by esthetic considerations that he undertook to bring this branch of mathematics into his gravitation theory *if at all possible*. It proved to be not just possible but startlingly successful, both esthetically and experimentally.

Einstein's equations

From the viewpoint of Riemannian geometry, there is a tensor that must *always* be present at every point in space, namely the metric tensor $g_{\mu\nu}$. This tensor establishes the geometry of space by giving a prescription for measuring the physical distance between points. If the geometry of space is just boringly Euclidean (in which case space and time together are said to be Minkowskian), then the metric tensor has everywhere the same uninteresting value

$$g_{\mu\nu} = \eta_{\mu\nu} \qquad (6)$$

where $\eta_{\mu\nu}$ (the "Minkowski metric") is the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

On the other hand, if spacetime is highly curved then the uniformity of Minkowski spacetime is lost, and the ones and zeros in $\eta_{\mu\nu}$ become functions of space and time in $g_{\mu\nu}$. Einstein knew that different objects fall with very precisely the same acceleration (Galileo's apocryphal expedition to the leaning tower having been supplanted by the exact experiments of Eötvös in 1889). This "equivalence principle" led him to suspect that the gravitational field should be somehow related to the geometry of space, so that the fall of bodies might be determined geometrically, without reference to their mass or composition. Thus, rather than build a theory with a separate gravitational field $h_{\mu\nu}$, Einstein's solution was to use the one metric tensor $g_{\mu\nu}$ as the gravitational field.

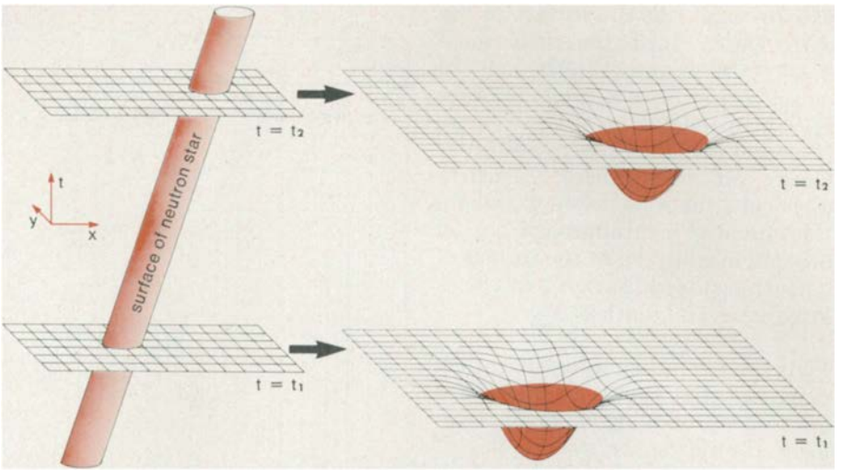


Figure 1. A neutron star moves through spacetime at constant velocity. To examine how the presence of matter curves the geometry of the space slices, we consider two snapshots (horizontal planes) at different times. If a coordinate grid is laid down through the star and then reassembled (embedded) in a Euclidean space, it will bulge out of the plane, as shown. The colored area of the bulge represents the interior of the neutron star. Note how the curvature in space moves with the star.

In the case where the gravitational field is very weak, one can say that our previous gravitational field $h_{\mu\nu}$ (satisfying eq. 5) is encoded into $g_{\mu\nu}$ as its difference from the Minkowski metric, that is,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad (7)$$

For the case where the deviation from flat space is great, an equation for $g_{\mu\nu}$ itself was devised, more complicated than eq. 5 but similar to it in character. That is, the full Einstein equation is still a wave equation but one that contains both "nonlinear" and "dispersive" effects. These effects cause a wave profile to undergo a change in shape as it propagates. Nonlinear processes can force the shape to steepen and even break (e.g. like ocean waves approaching a beach), whereas dispersive terms can flatten waves and lengthen their wavelengths. The physical process of the gravitational redshift is an example of this damping tendency in relativity. The question of whether steepening or flattening effects win in a particular situation is something that must be settled by numerical computer experiments, like those we will describe shortly.

Curved space

We can now take a closer look at the notion of curved space. In terrestrial situations the amount of curvature is quite small, and space is almost

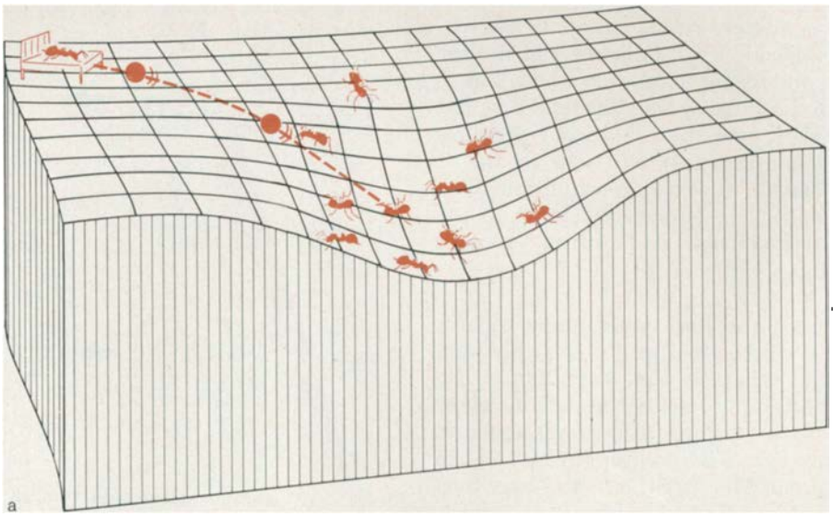
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Minkowskian. At the surface of the earth, the $h_{\mu\nu}$ due to the earth's mass is only $\sim 10^{-9}$, and even the mass of our galaxy induces $h_{\mu\nu}$ only of order $\sim 10^{-6}$. In general, an object of mass m and size r generates $h_{\mu\nu}$ of order Gm/rc^2 at most. There are astrophysical situations, however, where this quantity can approach one in order of magnitude. At the surface of a neutron star (pulsar), $h_{\mu\nu}$ can be as large as several tenths.

Figure 1 is a schematic "spacetime diagram" that shows the path of a neutron star moving at constant velocity through space, curving it as it goes. Spatial "snapshots" of the star's position are obtained by taking horizontal slices through the picture. Later times correspond to higher slices. (Of course the slice produces a 2-dimensional slice rather than a 3-dimensional space because not enough dimensions can be fitted into the diagram. We can suppose that each slice contains an equatorial cross section of the 3-dimensional star.) If a gridwork of meter sticks is assembled in the star's equatorial plane and is then reassembled far from the star's gravitational field (where space is Minkowskian), then the sticks will not lie in a plane! Rather, there will be a bulge that pops out of the plane, as shown schematically in the slices of Figure 1. The matter of the neutron star has warped its nearby space in this very direct sense.

Eddington (1928) first popularized the analogy of objects on a rubber sheet, each producing a dimple that in turn produces an attraction on other objects. If no mass-energy is present, the sheet is flat and the geometry is Minkowskian. A neutron star produces quite a large curved dimple (Fig. 1). Only after the work of Oppenheimer and Snyder (1939), and not fully until the 1960s, was an even more fantastic part of the analogy recognized: the elastic sheet of our space has a finite "tensile strength" (or, we can say, "elastic limit"). Stretched beyond this limit it will snap irreversibly into a kind of singular cusp which is a black hole. Eddington's analogy has been extended to this situation by Thorne (1967) in his Fable of the Ants, which we repeat here with some changes.

A family of ants who live on a rubber sheet decide to hold a reunion. All except one ant (who is ill) converge on the appointed spot. The ill ant keeps in touch by means of message balls that are projected out to him, rolling across the sheet. As the family comes together, the dimple in the rubber sheet becomes dangerously large, and the sheet starts stretching at a furious rate. (This slows down the message balls, and they arrive less and less frequently—the gravitational redshift.) At a critical point, the "elastic limit" of the sheet is exceeded and the dimple closes off into a bubble which shrinks, crushing the entire ant family into oblivion (see Fig. 2).



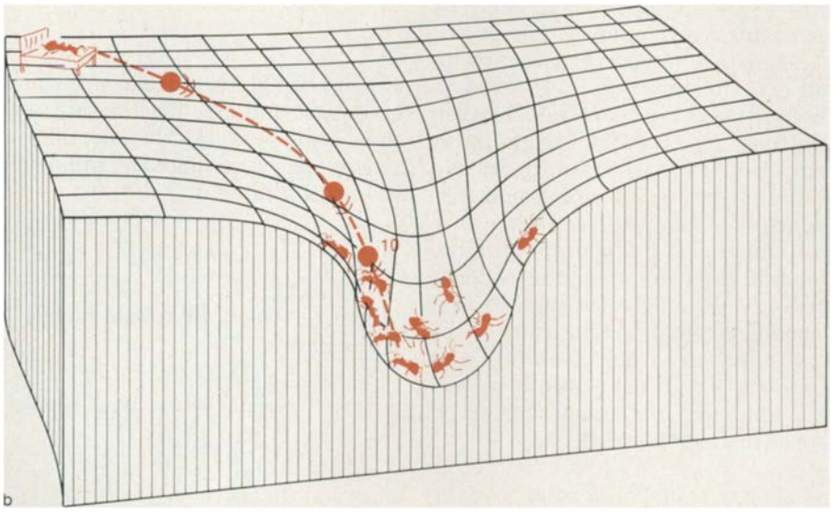
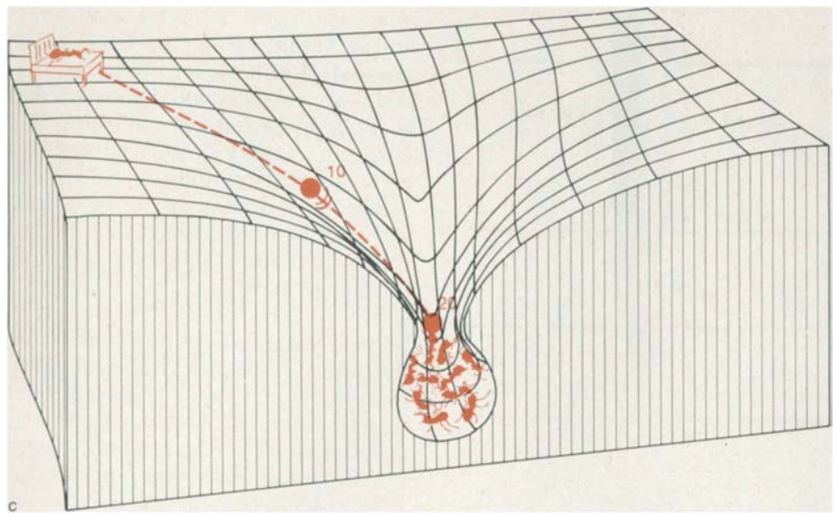
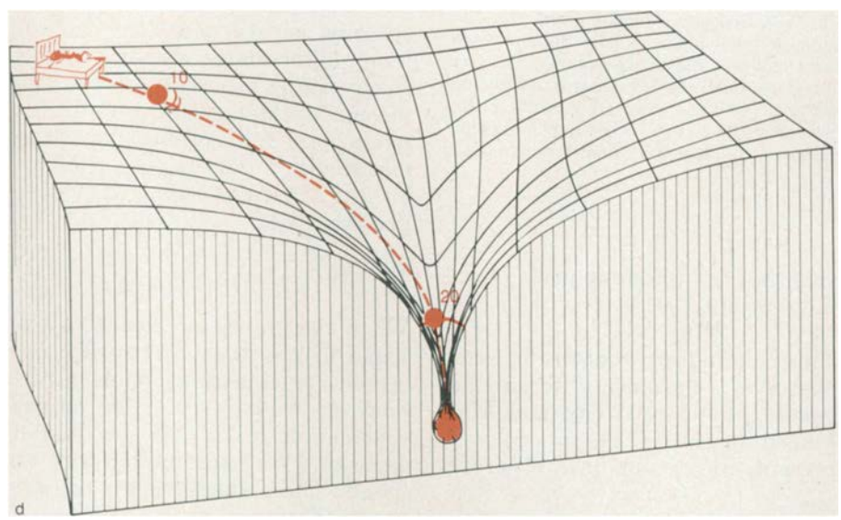


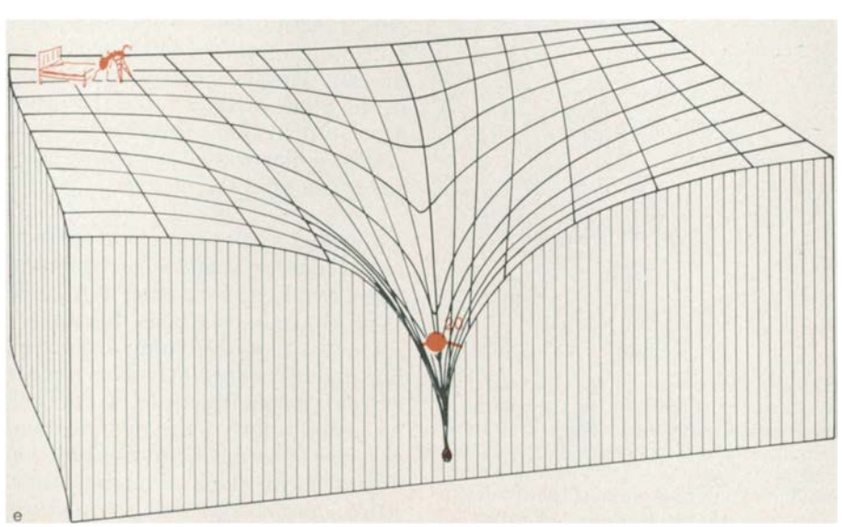
Figure 2. In the Fable of the Ants, adapted from Thorne (1967), the curved surface on which the ants live represents the curvature of space induced by the presence of matter (the ants). (*a*) The ants begin to assemble near the center of the rubber sheet. As they converge, message balls are rolled out to the ill ant far away. (*b*) The sheet begins to dip drastically when too many ants are too close together. Message ball 10 is slowed down by the surface slipping under it. (*c*) The ants are being dragged inside a black hole and message ball 20 is trapped at the surface of the hole. (*d*) The bag of ants is rushing toward zero volume and infinite density at the center of the black hole. (*e*) The singularity occurs, crushing the ant family. Message ball 20 is stuck forever rolling outward as fast as the surface slips below it. The ill ant has heard no news for so long that he sets off on crutches to investigate.

What does the single distant survivor think of this? He never finds out about it, because surrounding the singular cusp is a region (the interior of a black hole) that continues to stretch at furious velocity into the cusp. Message ball 10 in the figure (which said, "Ant Mabel fatter than

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ever; scenery seems oddly curved; wish you were here") is received only very late. Message ball 20 (which said "Help") is never received because it just balances forever, not quite outrunning the stretching sheet.

Although this fable may be whimsical, it actually captures most of the salient features of the gravitational collapse that forms a black hole—one of the most radically new predictions of general relativity. Figure 3, adapted from a review by Eardley and Press (1975), shows a spacetime diagram of a typical collapse scenario. If we imagine taking snapshots by slicing the spacetime, as in Figure 1, the spatial geometries obtained will be just those we have shown in Figure 2 (but without the ants). Instead of ants converging on the center, a star that has exhausted its nuclear fuel begins to collapse.

At early times (slice *a*, or Fig. 2a) the star makes a gentle dimple in the sheet because its density is fairly low and its radius large. Recall that Newton tells us that the gravitational force of attraction increases as the inverse square of the radius of the star, with the result that the collapse rapidly accelerates (slice *b*), quickly shrinking the star down to black hole dimensions (slice *c*). Any small deviations from perfect sphericity (rotation, magnetic fields, etc.) are amplified in relative size by the overall contraction of the star. These time-changing bumps make ripples in the geometry, which propagate outward at the speed of light and are called gravitational waves.

Finally, the black hole is born. The surface gravity on a black hole is so strong that even light cannot escape. The star continues to collapse inside the hole (slice *d*), but it is unable to communicate its fate to the outside world. In the end, the star's mass is contained in an arbitrarily small volume—a singularity. By this time all messages or gravitational waves have escaped to infinity, and a static black hole is left (slice *e*).

Curved time: Accelerated observers

Let us turn now to the more subtle notion of curved time. Just as the existence of a gravitational field implies a deviation from the uniform flatness of space, so it also implies a

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deviation from the uniform passage of time. The clocks of stationary observers near a massive body will tick more slowly than those of an observer far away; and the clocks of moving or freely falling observers will be different still. If we try to create a movie of a neutron star by assembling successive snapshots as single frames (as in Fig. 2), then in different parts of the movie screen time may be advancing at different rates. One part of the screen will be in "slow motion" relative to another. This phenomenon has become particularly important in the last few years as it has become possible to calculate the detailed time-dependent dynamics of strong gravitational fields by the use of fast computer systems. One of the key problems in this numerical work has been the choice of time intervals between snapshots—that is, to define time in a strong gravitational field.

In Minkowski space there is a naturally defined notion of time. This is the clock time that a set of observers "at rest" measure. Let us refer to these observers as the "Eulerian observers" of Minkowski spacetime. They are a *very* special class. First, they are unaccelerated, or inertial. The proper distance between any two of them is constant in time. Furthermore, the 3-space (3-dimensional space) defined by the moment when all the observers' clocks read the same value is geometrically flat. From a spacetime point of view, the trajectories or worldlines of these observers can be imagined as delineating the motion of an imaginary fluid. This fluid has no mass or energy associated with it since it is only a "coordinate fluid." In Minkowski spacetime, the fluid can be irrotational, incompressible, and nonshearing.

Not so in curved space of a gravitational field! If we try to use freely falling, unaccelerated observers in a gravitating spacetime, the observers fall toward whatever centers of attraction are around. Thus the coordinate fluid (which will become the grid mesh of a computer program) is subject to distortions over which we have little control. We can correct this situation only by having each of our Eulerian observers accelerate (fire their rocket engines) enough to balance the gravitational force. However, this acceleration causes time dilation effects, so we are back to the problem of how to synchronize the advance of snapshots in different parts of the movie frame.

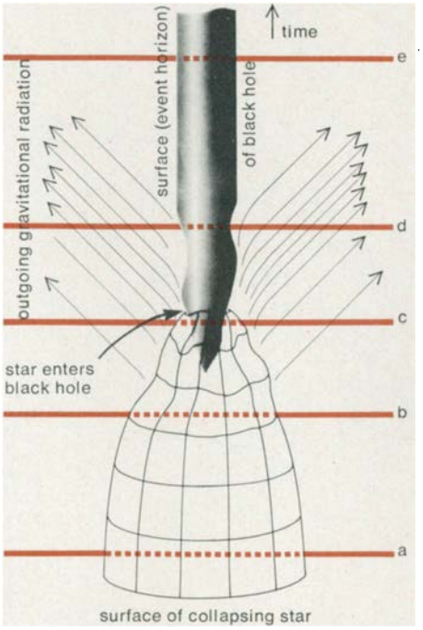


Figure 3. Spacetime diagram of a star that collapses to form a black hole. Spatial "snapshots" taken by slicing this picture at the times a,b,c,d,e would have the same spatial curvature as the successive surfaces in Figures 2*a–e*. The mass of ants in Figure 2 is here replaced by the mass of a star that has exhausted its nuclear fuel.

This nonuniqueness of the Eulerian observers is a primary difference between general relativity and Minkowski spacetime. When gravity is present, we cannot have observers who are *both* unaccelerated and static. Therefore we have to choose a new set of criteria for observers "at rest," whose clocks will define how we slice the spacetime into snapshots. In Einstein's theory space is curved directly by gravity, and time is curved in the sense that it becomes rather arbitrary how one should advance the time coordinate at one point in space relative to another.

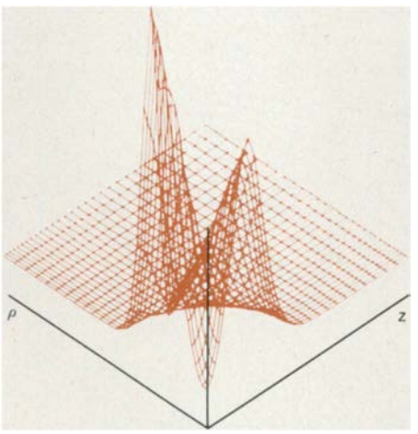


Figure 4. The curvature of space due to a cloud of gravitational radiation. At the moment of maximum density shown here, there are no wavelike features noticeable. The curvature dips at the origin ($\rho = z = 0$) because we have multiplied it by r, the distance from the origin. The axes ρ and z label points on the grid.

Because we are more interested in the dynamics of the gravitational field than in the motion of the observers through the field, it seems most natural to choose accelerated observers who are as stationary as possible. In 1944, Lichnerowicz first proposed what has since become the "industry standard." The acceleration is chosen so that the observer fluid acts exactly like an *incompressible* fluid. This means that volume-preserving shearing can still occur, but the Eulerian observers do not converge as freely falling observers would. The incompressible specification of Eulerian observers is often called *maximal slicing*, since the volume of the 3-slices is maximal compared to other possible slices.

Evolving spacetimes

When the gravitational field is dynamical, the distance between observers is time-dependent. Since, however, we have fixed the volume contraction of our observer fluid at zero, only shearing motions are allowed. These shearing motions are the fundamental modes of gravitational radiation. As a gravitational wave passes by a set of masses, their relative positions first stretch in one direction and contract in the orthogonal direction, and then (on the next half-cycle of the wave) reverse the process. The shearing occurs entirely in the plane orthogonal to the propagation direction. Thus we say that gravitational waves are *transverse-shear waves* in the metric.

To evolve a spacetime, we pick an initial time $t = 0$, and at each point in space we place an observer at rest. He then continually tunes his rocket motor so that the volume between him and his nearest neighbors neither expands nor contracts. In his log book he keeps track of the distance and shearing of his neighbors as a function of his local clock-measured time. After a long enough time, when things

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have settled down to a final static state, all the log books are collected. From an audit of the books we can recover the curved geometry as a sequence of snapshots—these in turn tell all there is to know about the gravitational field.

This technique for building up a spacetime from a given set of initial data has been translated into a computer program. Over the last few years the program has successfully evolved a number of dynamic spacetimes. Starting with Minkowski spacetime, one can evolve gravitational fields of small amplitude and compare them to known analytic solutions of the linearized Einstein equations (our eq. 5 above).

Building confidence from these checks, one can then turn to strongly nonlinear problems, perhaps best exemplified by the collision of two black holes. The black hole example is important for both relativity and astrophysics. It is a typical one in a class of situations in which strongly gravitating objects interact in their "near zone" to produce ripples that appear as gravitational radiation in the "far zone." There exists no other mathematical technique for investigating these dynamic processes in general relativity than the use of high-speed computers. We now review the actual numerical results for gravitational waves and colliding black holes and see how well our "rubber sheet" intuition works.

For a weak-field example, we can go back to the Fable of the Ants and change the ending to a happier one. Instead of assembling at their catastrophic reunion, the ants all stand off at some distance. By stamping up and down they generate ripples in the rubber sheet, which represent gravitational waves. These waves converge toward the reunion site at the speed of light. At their moment of maximum convergence (which we can call $t = 0$) they form a cloud of radiation which gives a gentle broad depression to the sheet. The waves then pass through each other and the cloud breaks up into outgoing ripples that eventually disperse to infinity.

There is no catastrophe here because the amount of energy in the waves is very small. This system could have been described perfectly well by using $h_{\mu\nu}$ (eq. 5 with $T_{\mu\nu} = 0$) instead of $g_{\mu\nu}$.

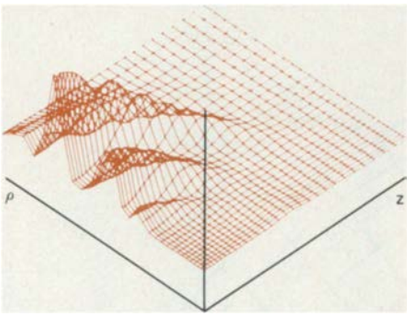


Figure 5. The same cloud of radiation as in Figure 4 is shown at a later time. The curvature is now a series of outgoing waves with no curvature left near the origin. Note that the curvature is greatest on the ρ axis and falls off rapidly toward the z-axis—a characteristic of quadrupolar gravitational radiation.

This fact makes it possible to work out the evolution of the ripples analytically (i.e. without a computer), as Teukolsky recently did, starting from the initial "snapshot" that was first studied by Brill and Wheeler in the early 1960s. The next step was to try this "test example" on the complicated computer program. This was done by Eppley and Smarr in 1976.

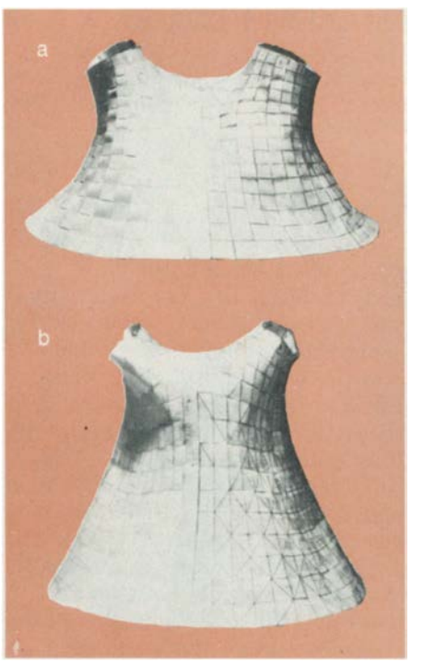


Figure 6. These "straw hats for a horse"—embedding diagrams (analogies of Fig. 2) for two colliding black holes—show the proper distances and angles between grid lines. (*a*) At the initial instant the two holes are at rest rather close to one another. The open ends are slightly inside of where the surface of the holes were initially. Note that the holes are already curving the surrounding space substantially. (*b*) After the collision of the two holes, the bag has stretched out much more. The new location of the surface of the last black hole is somewhere on the long neck. The grid squares have been deformed by the motion of the holes.

Figure 4 shows the curvature induced by the wave at $t = 0$. The value of the curvature has been multiplied by the distance from the origin r, so that waves which fall off as $1/r$ will stand out as the evolution progresses. Because of this factor, the graph is pulled down at $r = 0$, but otherwise it would rise smoothly to its highest value at the origin. As time progresses, the peak drops rapidly and splits into a series of outgoing pulses (Fig. 5). Note that the amplitude of the wave is greatest on the ρ axis (which is the equatorial plane) and falls off quickly toward the z-axis (the pole). This reflects the fact that the fundamental mode of gravitational wave emission is *quadrupole* (the same kind of directionality that a circular-loop FM auto antenna has).

The grid in Figures 4 and 5 is the actual spacing of mesh points used by the computer in the calculation. One fundamental rule must be applied to an evolution like this—namely, the ratio of the wavelength of the radiation to the spacing of the mesh points must be much larger than one. Here the ratio is between 5 and 10. The calculations of these figures were done on the CDC 7600 computer at Lawrence Livermore Laboratory, and they took about 15 minutes of computer time.

A featureless depression on our rubber sheet, then, can become a series of outgoing waves. When we view the computer evolution not as a series of snapshots but as an actual movie made by Eppley and Smarr in 1976, we are struck by how much the process resembles an actual rubber sheet being pulled down and released at $t = 0$. The membrane oscillates up and down, sending out a series of waves, and it finally comes to rest in a flat configuration. The analogy of spacetime as an elastic medium becomes very real.

Strong fields—colliding black holes

The same computer program that evolves these weak gravitational waves can also evolve the initial slices of much stronger waves. Indeed, the $t = 0$ solutions of Brill can be turned

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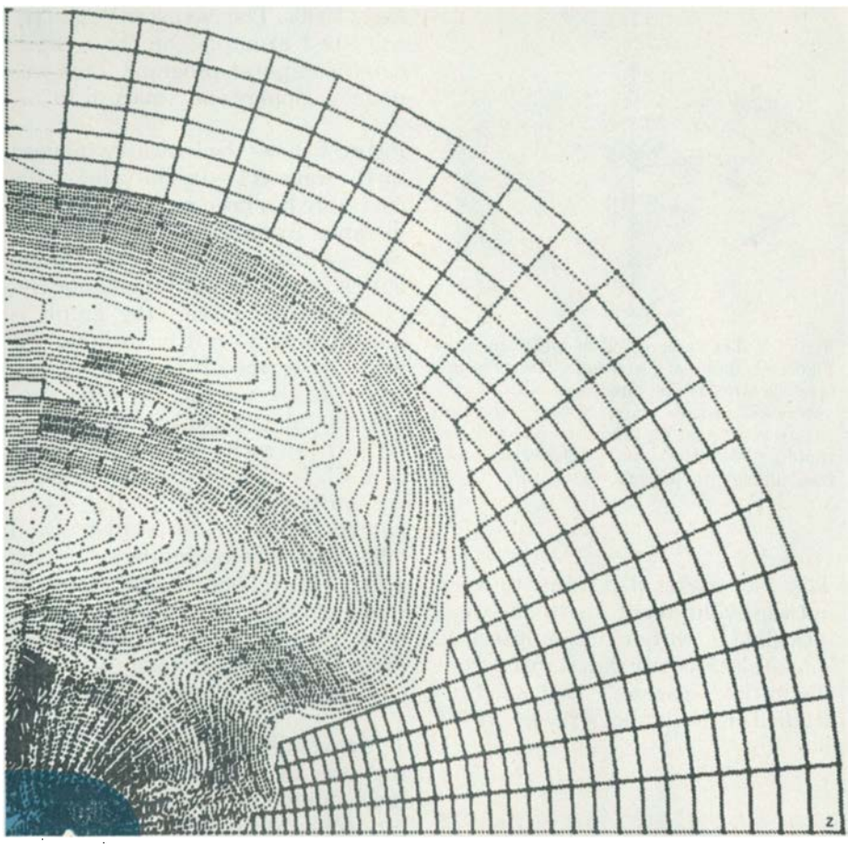


Figure 7. The colliding holes of Fig. 6 sent out ripples of gravitational radiation. Here a contour map of the curvature is shown after the collision. Just as in Fig. 5, there are several pulses with their maximum values on the z-axis. The final hole is shown in color. Note that the wavelength of the radiation is large compared to the holes, because of the gravitational redshift.

up in strength arbitrarily. As the amplitude is increased, the size of the initial cloud's metric perturbations $h_{\mu\nu}$ approaches unity, and the mass of the cloud strongly curves the surrounding space. Here the concept of linearizing off a Minkowski background has no meaning, since the "background" is dominated by the wave itself. The evolution of such a cloud is both nonlinear and dispersive. If the amplitude is finally pushed sufficiently high, then the cloud of gravitational radiation finds itself inside a black hole created by its own mass—a black hole made of nothing but curved space! This is how nonlinear Einstein's theory can be.

Regardless of how black holes form, whether "mathematically" as above, or more physically by the process of stellar collapse, we may want to study their interactions after the formation process has occurred. The simplest scenario that one can imagine for study is the head-on collision of two nonrotating black holes. This "two-body problem" in general relativity is vastly more difficult than the corresponding two point-mass problem in Newtonian gravity, just because Einstein's gravity is a *dynamical* field theory (and nonlinear at that). Further complications arise because of the identification of gravity with metric distance. For instance, it is very difficult in general relativity to answer such simple questions as What is the distance between the two black holes at time t?, because of the curved nature of space and time. So we must again turn to the computer.

Of course, this is no cause for despair. It marks, in some sense, the maturing of general relativity as a discipline of "realistic" physics. A number of other fields in physics, such as hydrodynamics, aerodynamics, and plasma physics, have passed through stages where the important processes could be elucidated analytically, to a stage where most significant problems require numerical calculations. General relativity is just now arriving at this stage of development.

To model the collision of two black holes, we will assume that two black-hole "cusps" have already formed on our rubber sheet at $t = 0$. They then start falling toward each other, accelerating as the force of gravity between them increases. Their dynamical motion causes ripples to propagate outward in the sheet. If these waves start early enough, they can escape, but as the time of collision of the two black holes nears, any would-be ripples are sucked back into the final black hole formed from the merger of the two initial ones. This single black hole will finally "shake off" its remaining nonsphericity by generating gravitational waves in a decaying "ringing" mode. Spacetime dynamics is on center stage here!

To visualize the computer snapshots of this collision, we will use the grid of meter sticks that we used earlier to show the curved equatorial plane of a neutron star—embedded (i.e. reassembled) in a 3-dimensional space where gravitational fields are not so dominant. Since the collision of two black holes is symmetric around the line of centers of the two holes, we need to display the geometry only in one plane containing this axis and the holes. In this plane we lay down our network of grid lines and mesh points labeled by cylindrical coordinates ρ and z. For any desired snapshot, the computer will print out the log books giving the "proper length" of each grid line segment and the angles at each corner of the mesh quadrilateral.

Now how to piece these quadrilaterals together into the 3-dimensional embedding diagram? One might suppose that the computer could do this too and could draw a picture of the resultant surface as seen from any perspective. Indeed, it is easy to devise computer programs which do this for simple embedding diagrams like Figure 1. In the general case, however, it turns out to be very difficult for a computer to construct a good embedding model. The reason is that even when the shapes of all the coordinate quadrilaterals are specified, there is still some flexibility in the model, corresponding to taking a pliable paper model and crumpling it

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between one's hands. It is not so easy to describe mathematically the constraint that our model should be "minimally crumpled." A computer embedding program written by Andrej Čadež (1975) had some success, but similar approaches also turned out pictures in which a surface would curve smoothly in one part of the picture, only to degenerate into a crumpled mess in another part.

Bryce DeWitt and his students invented a different approach. First, the computer draws each individual quadrilateral on a flat piece of paper (this is easy). Then, on sufficient advance notice, a large quantity of beer, food, scissors, scotch tape, and graduate students are assembled. The graduate students cut the squares out and tape them into strips. Other students then weave the strips together, basket fashion, into a final embedding diagram. A sample result is shown in Figure 6. Only the region of strong curvature is shown here. The two holes at the top represent the surface of the initial two black holes. Because we are not so interested in what goes on inside the black holes, we have cut off that portion of the surface. Among graduate basketweavers, this kind of embedding diagram is called a "horse's hat," because of the obvious resemblance. (Incidentally, the DeWitt technique for constructing embedding diagrams lends some credence to the complaint that graduate school these days is only so much advanced basketweaving.)

The top horse's hat in Figure 6 represents the spatial curvature at $t = 0$. Here the holes are at rest, with a separation several times their own size. The lower "hat" shows the slice after the holes have collided. The neck surrounding both of them has stretched and is forming a "bag" containing what had been the two holes. The surface of the new, single hole has formed at about the constricted region in the long neck. Notice how the grid squares have been pulled into longer rectangles or squashed into parallelograms. If we followed the evolution even further, the neck would stretch more and more, cutting off the bag from the rest of the sheet.

What gravitational ripples are generated by this collision? One way to view this is by a contour plot, not of the ripples themselves but of that part of the curvature that carries momentum. Such a plot, made at about the time of collision, is shown in Figure 7 (from the work of Eppley and Smarr). Note that there are several peaks in the outgoing curvature that look very similar to the peaks of the pure gravitational wave in Figure 5. The final black hole can be seen to be much smaller than the distance between wave peaks. This indicates that the waves formed have been redshifted, or lengthened, as they climb the steep potential "hill" to get away from the hole.

So the old Eddington idea of a rubber-sheet spacetime is not very far off, even for the collision of two black holes. The holes (very dense ants) warp the sheet and attract each other, and they fall together into a bag that stretches and cuts itself off from the world. Instead of message balls, ripples of curvatures—gravitational waves—propagate outward to inform us—or the ill ant—of the catastrophic collapse that has taken place. Now that a computer program has been written to follow the detailed shearing and stretching of the spacetime fabric, it will be used to investigate many other strong-field problems: nonspherical star collapse, formation of supermassive black holes, and spiral collisions of neutron stars or black holes. Each such event will send out its characteristic message, awaiting future observation by gravitational wave astronomers.

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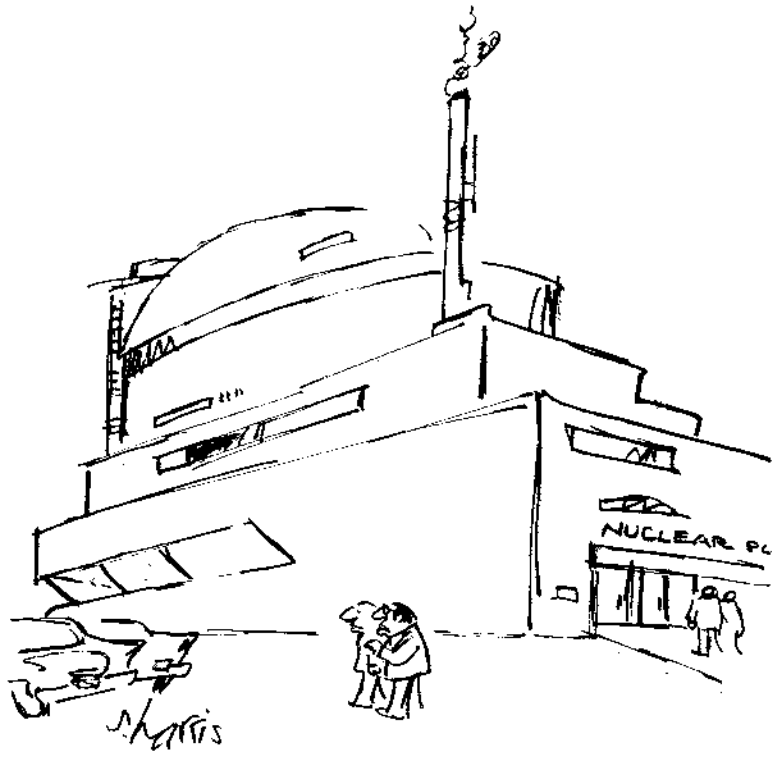
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"A black hole—that's the answer to our radioactive waste problem."

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