Shedding Light on Black Holes

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1. Introduction

Black holes are among the most enigmatic objects in the physical universe. The fact that they elude direct observation by not allowing light to escape from their surface and that they are able to warp space and time into bizarre topologies has made them a favorite subject of science fiction writers. Although they may be the source of countless fantasies, they also may happen to be the source of a large number of realistic astrophysical phenomena.

The black hole emerged as one of the first (1916) non-Newtonian "objects" to be recognized in Einstein's General Theory of Relativity. Einstein's theory holds that gravity is not a force, but a curvature of space. Thus, in the solar system, the earth goes around the sun not because the sun "pulls" on the earth, but because the sun's mass-energy warps the surrounding space into a "bowl" shape. The earth then moves in a "straight line' (geodesic) in the curved space around the rim of the bowl. If a light ray moves near the sun, it will be deflected from a Euclidean straight line by the same spatial curvature. Furthermore, a clock ticks more slowly in a deep gravitational well. Thus, time itself is effected, as well as space, by relativistic gravity.

If one imagines the well getting deeper and deeper, for instance caused by a stellar core collapse, then at a certain point the curvature will become so strong that even light itself cannot pull out of the curved space. The spherical surface where this occurs is called the "event horizon" or surface of the black hole. The points inside this surface are causally cut off from the rest of the universe. That is the event horizon is a one-way surface through which things may enter the black hole, but once having done so they can never return or even influence events occurring in the exterior of the hole.

Astrophysicists have invoked the strong gravitational fields near black holes to explain energetic phenomena that otherwise have eluded explanation. Quasars (the most luminous objects in the universe), active galactic nuclei (including our own milky way), and galactic x-ray sources (such as the famous Cygnus X-1) are all believed to be powered by matter falling down the deep potential wells of black holes. However, in spite of the success of the models in explaining the observational data, there is still no direct observational evidence for the existence of black holes. If we are lacking direct evidence of the existence of black holes, then why do we go to so much effort to study them?

First there are the theoretical reasons. General relativity provides us with a description of gravitation in terms of the curvature of space and time. It is a beautiful geometric theory and as the culmination of the genius of Einstein, one of the great achievements of twentieth century science. Black holes are the fundamental "bodies" that exist in general relativity. That is, they are the simplest vacuum gravity solutions to the Einstein field equations that are stable. Two black holes orbiting each other or colliding is the "two-body problem" of general relativity. However, because such motions alter the spacetime geometry in a nonlinenar dynamical fashion, this seemingly simple problem is as yet unsolved. It seems likely that it can only be solved numerically.

General relativity is also the successor to Newton's theory of the gravitational field and, try as many physicists have to come up with a better theory, Einstein's description has weathered experimental test after experimental test [1]. In spite of the number of observations (almost all based on weak gravitational field) that have confirmed general relativity as the correct theory of gravity, there are a number of predictions of general rela-

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tivity that have remained as yet untested. The two most significant of these are the existence of gravitational radiation and the existence of black holes. Although both of these phenomena were mathematically described in the second decade of this century, they have remained hidden from all *direct* experiments and observations even to this very day. Thus, the need to study black hole dynamics, which should be one of the strongest sources of gravitational radiation, to provide new experimental tests of the validity of strong field general relativity.

Black holes may also provide the means to link quantum mechanics, thermodynamics and general relativity together in one complete theory. One of the great challenges in theoretical physics today is to construct a quantum theory of gravity, and many researchers are now studying the behavior of black holes and their effects on quantum fields, hoping that the secrets of quantum gravity will be revealed. It has also been demonstrated that black holes obey laws similar to the laws of thermodynamics. Quantum black holes, should they exist, may also be remnants of the formation of the universe and their detection may lead to a better understanding of the nature of the cosmos. A popular account of the quantum mechanics of black holes can be found in the recent best seller written by Steven Hawking [2].

From a more 'practical' point of view, black holes are important to the numerical astrophysicist interested in providing detailed explanations for the processes that are thought to power some of the most violent of astrophysical events. Black holes are expected to have formed from the collapsed cores of galaxies, quasars, and star clusters. These black holes may have masses up to hundreds of millions times that of our sun. In some of these cores, the black holes may still be in the process of swallowing up vast amounts of matter in the form of stars, dust and interstellar gases. Thus, one wishes to solve for the collapse process which created the hole and then for the accretion processes which later fuel the holes.

Black holes are also thought to occur from the collapse of single stars that have reached the end of their nuclear evolutionary life time. Since a large majority of stars belong to binary stellar systems, it is expected that black holes themselves will also often be members of binary systems, after one of the stars collapses to form a hole. If the normal companion star is in close proximity to the black hole, the strong gravitational field of the latter can strip off part of the atmosphere of the former and thereby copiously produce x-ray radiation as the stellar matter accelerates rapidly toward the surface of the black hole. Strong x-ray sources have been observed by satellite observatories specifically designed to detect such systems and there is strong evidence that some of these sources may in fact result from binary systems with one member being a black hole, the Cygnus X-1 system being the best studied.

If the phenomena described above actually occur, then astrophysicists need to know the details of their dynamics. The equations of general relativity consist of a complicated set of nonlinear coupled partial differential equations for both gravitational field and the matter that interacts with the field. Attempts at trying to understand the behaviour of these equations in the presence of strong gravitational fields and velocities close to the speed of light using analytic means have led to limited success. It would seem that supercomputer simulation will be the only tool powerful enough to provide a "numerical laboratory" for studying both the nature of strong field dynamics in general relativity and for building detailed astrophysical models of black holes interacting with matter. Because of the necessity of using computational methods to make progress in this fundamental set of problems, studying black holes numerically is clearly one of the Grand Challenges in computational science.

2. Numerical Relativity

The systems being studied by numerical relativists can be as varied as the formation of neutron stars during supernova explosions, the collision of neutron stars or black holes, or the formation and evolution of the universe itself. In the remainder of this paper we will concentrate on black hole physics, but for those interested in the state-of-the-art of numerical relativity and all that it has to offer, it is recommended that the reader consult the proceedings of a conference on numerical relativity recently (May, 1988) held at the National Center for Supercomputing Applications (NCSA) [3]. Because of the availability of this lengthy and comprehensive work, our paper

will not attempt completeness in referencing either the historical or current research in the field, but rather give an overview of the computational challenges which the field presents.

3. Spacetime to Space plus Time

The basis of most studies in numerical relativity can be formulated in the so-called 3 + 1 decomposition of spacetime. Although the theory of general relativity treats space and time on an equal footing, the 3 + 1 formalism splits the four dimensional spacetime into a time series of three-dimensional curved spatial hypersurfaces called "time slices." In addition, time can evolve at different rates at each point in space and these rates can vary from slice to slice. Therefore, unlike most other fields of computational physics, the geometry of space and time must be solved for and is not given as a Euclidean background. The spatial geometry of each time slice is specified by a "3-metric" which is a symmetric tensor consisting of six functions. The time variation at each point of the time slice is given by a scalar "lapse function." Finally, the spatial coordinates may move from slice to slice, and the 3-velocity of the coordinates, relative to observers at rest in the time slice, is specified by the "shift vector" which contains three functions. All ten of these functions, which together determine the full 4-metric of spacetime, are, in the general case, dependent on three spatial coordinates and a time coordinate. For a comprehensive treatment of the 3 + 1 approach see [4].

The equations which result from projecting the Einstein equations into 3 + 1 form are typically six hyperbolic equations for the 3-metric variables coupled to four elliptic equations for the lapse and shift functions. In addition, the Einstein equations are a constrained system, so that there are four elliptic equations for constrained variables on each time slice (see York [5]). One can often "trade" some of the hyperbolic equations for some of these elliptic equations. However, the coefficients of terms in all the equations depend on the variables for which these equations solve, so the elliptic-hyperbolic system is tightly coupled.

Because of the strong nonlinearities in the Einstein equations, curvature of space and time can arise without any matter or source fields. Such "vacuum" spacetimes can include gravitational radiation, black holes, or even closed cosmologies. In this case, the Einstein equations can be studied as the ultimate in dynamical nonlinear field theories, with relativists sharing many of the goals of researchers who study solitons, chaos, and strange attractors. Since the solutions to the 3 + 1 Einstein equations can be looked upon as representing a one parameter family of curved 3-geometries, numerical relativists also have a strong overlap with mathematical geometers, who are themselves relying more and more on supercomputers and visualization.

When matter fields are present, one must solve the fully general relativistic equations of matter flows. These can include gas dynamics, magneto-hydrodynamics, electromagnetic or neutrino coupled hydrodynamics, or n-body dynamics. The major new physical effects are caused by special relativistic length contractions or time dilations, the replacement of the scalar gravitational field potential by a tensor potential, and the time dilation effect of strong gravitational fields. Typically, numerical relativists try to base their solutions of the matter equations on the techniques used in nonrelativistic cases.

In summary, one can state the physics of relativistic gravitating matter, paraphrasing John Wheeler, as follows: "Mass-energy (including the nonlinear excitations of the gravitational field itself) tells the spacetime how to curve and the curved spacetime tells the mass-energy how to move." In the following sections, we will outline how one uses high performance computing devices to take this general dictum and treat specific cases.

4. Symbol Crunching

One of the major hurdles that must be overcome in numerical relativity is the accurate coding of the complicated expressions that appear in the Einstein equations. Analytic solutions of the Einstein equations have been restricted to special cases that invoke symmetries to simplify these coupled nonlinear differential equations to manageable form. However, even under these circumstances symbolic computation methods have been standard practice for over a decade. Since the equations for dynamic, asymmetric spacetimes are even more complex, it is clear that use of

SPACETIME METRIC

$$g_{\mu\nu} = \begin{bmatrix} -\alpha^2 & 0 & 0 & 0 \\ 0 & a\psi^4 & c\psi^4 & 0 \\ 0 & c\psi^4 & b\psi^4 & 0 \\ 0 & 0 & 0 & d\psi^4\sin^2(\xi) \end{bmatrix}$$

define: $q = ab - c^2$

$$R_{12} = 0 = -\frac{2ac\frac{\partial\psi}{\partial\xi}\cot(\xi)}{\psi q} + \frac{2ab\frac{\partial\psi}{\partial\eta}\cot(\xi)}{\psi q} - \frac{\frac{\partial a}{\partial\xi}c\cot(\xi)}{2q} + \frac{a\frac{\partial b}{\partial\eta}\cot(\xi)}{2q} - \frac{2\frac{\partial\psi}{\partial\eta}\cot(\xi)}{\psi} - \frac{\frac{\partial d}{\partial\eta}\cot(\xi)}{2d} - \frac{2ac\frac{\partial^2\psi}{\partial\xi^2}}{\psi q}$$

$$- \frac{2ac\left[\frac{\partial\psi}{\partial\xi}\right]^2}{\psi^2 q} - \frac{4c^2\frac{\partial\psi}{\partial\eta}\frac{\partial\psi}{\partial\xi}}{\psi^2 q} + \frac{8ab\frac{\partial\psi}{\partial\eta}\frac{\partial\psi}{\partial\xi}}{\psi^2 q} - \frac{ac\frac{\partial d}{\partial\xi}\frac{\partial\psi}{\partial\xi}}{d\psi q} + \frac{ab\frac{\partial d}{\partial\eta}\frac{\partial\psi}{\partial\xi}}{d\psi q} - \frac{2a\frac{\partial c}{\partial\xi}\frac{\partial\psi}{\partial\xi}}{\psi q} + \frac{8c\frac{\partial c}{\partial\eta}\frac{\partial\psi}{\partial\xi}}{\psi q} - \frac{2a\frac{\partial\alpha}{\partial\xi}c\frac{\partial\psi}{\partial\xi}}{\alpha\psi q}$$

$$- \frac{3\frac{\partial a}{\partial\xi}c\frac{\partial\psi}{\partial\xi}}{\psi q} - \frac{a\frac{\partial b}{\partial\eta}\frac{\partial\psi}{\partial\xi}}{\psi q} + \frac{2a\frac{\partial\alpha}{\partial\eta}b\frac{\partial\psi}{\partial\xi}}{\alpha\psi q} - \frac{2\frac{\partial a}{\partial\eta}b\frac{\partial\psi}{\partial\xi}}{\psi q} - \frac{2bc\frac{\partial^2\psi}{\partial\eta^2}}{\psi q} + \frac{4c^2\frac{\partial^2\psi}{\partial\eta\,\partial\xi}}{\psi q} - \frac{2bc\left[\frac{\partial\psi}{\partial\eta}\right]^2}{\psi^2 q}$$

$$+ \frac{ab\frac{\partial d}{\partial\xi}\frac{\partial\psi}{\partial\eta}}{d\psi q} - \frac{bc\frac{\partial d}{\partial\eta}\frac{\partial\psi}{\partial\eta}}{d\psi q} - \frac{2b\frac{\partial c}{\partial\eta}\frac{\partial\psi}{\partial\eta}}{\psi q} - \frac{3\frac{\partial b}{\partial\eta}c\frac{\partial\psi}{\partial\eta}}{\psi q} - \frac{2\frac{\partial\alpha}{\partial\eta}bc\frac{\partial\psi}{\partial\eta}}{\alpha\psi q} + \frac{2a\frac{\partial b}{\partial\xi}\frac{\partial\psi}{\partial\eta}}{\psi q} + \frac{2a\frac{\partial\alpha}{\partial\xi}b\frac{\partial\psi}{\partial\eta}}{\alpha\psi q} + \frac{3\frac{\partial a}{\partial\xi}b\frac{\partial\psi}{\partial\eta}}{\psi q}$$

$$- \frac{a\frac{\partial b}{\partial t}\frac{\partial c}{\partial t}\psi^4}{4\alpha^2 q} - \frac{\frac{\partial a}{\partial t}b\frac{\partial c}{\partial t}\psi^4}{4\alpha^2 q} + \frac{\frac{\partial a}{\partial t}\frac{\partial b}{\partial t}c\psi^4}{2\alpha^2 q} - \frac{\frac{\partial a}{\partial\xi}c\frac{\partial d}{\partial\xi}}{4dq} + \frac{a\frac{\partial b}{\partial\eta}\frac{\partial d}{\partial\xi}}{4dq} - \frac{\frac{\partial b}{\partial\eta}c\frac{\partial d}{\partial\eta}}{4dq} + \frac{\frac{\partial a}{\partial\xi}b\frac{\partial d}{\partial\eta}}{4dq} + \frac{\frac{\partial c}{\partial\eta}\frac{\partial c}{\partial\xi}}{q} - \frac{\frac{\partial a}{\partial\xi}\frac{\partial c}{\partial\xi}}{2q}$$

$$+ \frac{c\frac{\partial^2 c}{\partial\eta\,\partial\xi}}{q} - \frac{\frac{\partial b}{\partial\eta}\frac{\partial c}{\partial\eta}}{2q} - \frac{\frac{\partial^2 b}{\partial\eta^2}c}{2q} - \frac{\frac{\partial\alpha}{\partial\eta}\frac{\partial b}{\partial\eta}c}{2\alpha q} - \frac{\frac{\partial a}{\partial\xi}\frac{\partial\alpha}{\partial\xi}c}{2\alpha q} - \frac{\frac{\partial^2 a}{\partial\xi^2}c}{2q} - \frac{\frac{\partial a}{\partial\eta}\frac{\partial b}{\partial\xi}}{2q} + \frac{a\frac{\partial\alpha}{\partial\xi}\frac{\partial b}{\partial\eta}}{2\alpha q} + \frac{\frac{\partial a}{\partial\xi}\frac{\partial b}{\partial\eta}}{2q}$$

$$+ \frac{\frac{\partial a}{\partial\xi}\frac{\partial\alpha}{\partial\eta}b}{2\alpha q} - \frac{4ac^2\frac{\partial c}{\partial\xi}\frac{\partial\psi}{\partial\xi}}{\psi q^2} + \frac{2a^2 b\frac{\partial c}{\partial\xi}\frac{\partial\psi}{\partial\xi}}{\psi q^2} - \frac{8c^3\frac{\partial c}{\partial\eta}\frac{\partial\psi}{\partial\xi}}{\psi q^2} - \frac{6abc\frac{\partial c}{\partial\eta}\frac{\partial\psi}{\partial\xi}}{\psi q^2} - \frac{3a\frac{\partial b}{\partial\eta}c^2\frac{\partial\psi}{\partial\xi}}{\psi q^2} - \frac{3\frac{\partial a}{\partial\eta}bc^2\frac{\partial\psi}{\partial\xi}}{\psi q^2}$$

$$+ \frac{a^2\frac{\partial b}{\partial\xi}c\frac{\partial\psi}{\partial\xi}}{\psi q^2} + \frac{a\frac{\partial a}{\partial\xi}bc\frac{\partial\psi}{\partial\xi}}{\psi q^2} + \frac{2a^2 b\frac{\partial b}{\partial\eta}\frac{\partial\psi}{\partial\xi}}{\psi q^2} + \frac{2a\frac{\partial a}{\partial\eta}b^2\frac{\partial\psi}{\partial\xi}}{\psi q^2} + \frac{2abc\frac{\partial c}{\partial\xi}\frac{\partial\psi}{\partial\eta}}{\psi q^2} - \frac{4bc^2\frac{\partial c}{\partial\eta}\frac{\partial\psi}{\partial\eta}}{\psi q^2}$$

$$+ \frac{2ab^2\frac{\partial c}{\partial\eta}\frac{\partial\psi}{\partial\eta}}{\psi q^2} + \frac{a\frac{\partial b}{\partial\xi}c^2\frac{\partial\psi}{\partial\eta}}{\psi q^2} + \frac{\frac{\partial a}{\partial\xi}bc^2\frac{\partial\psi}{\partial\eta}}{\psi q^2} + \frac{ab\frac{\partial b}{\partial\eta}c\frac{\partial\psi}{\partial\eta}}{\psi q^2} + \frac{\frac{\partial a}{\partial\eta}b^2 c\frac{\partial\psi}{\partial\eta}}{\psi q^2} - \frac{2a^2 b\frac{\partial b}{\partial\xi}\frac{\partial\psi}{\partial\eta}}{\psi q^2}$$

Fig. 1. The hyperbolic equation which governs the time evolution of metric component "c" in an axially symmetric spacetime

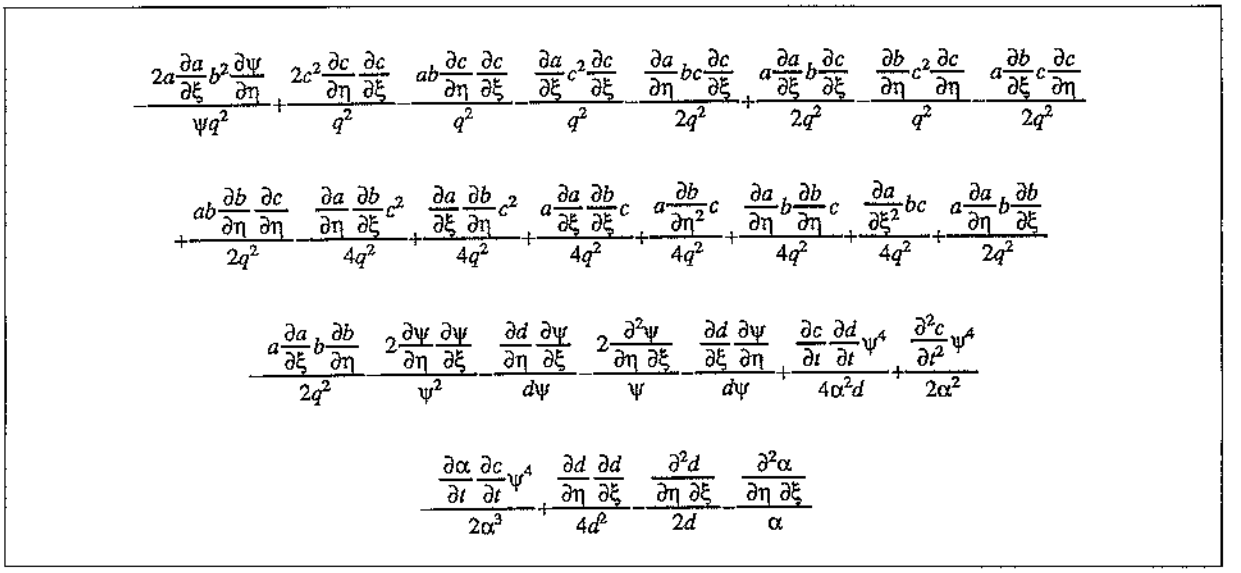


Fig. 1. (continued).

symbolic manipulation packages are going to be critical to numerical relativity.

As an example of the difficulties occurring in deriving the expressions to be entered into the numerical code, we look at one of the equations which arises in our black hole code. *Fig. 1* shows one of the hyperbolic equations that must be solved for a three metric component "$c(\eta, \xi, t)$" in the special case of axial and reflection symmetry. We use spherical spatial coordinates (radial coordinate η, polar coordinate ξ, and azimuthal coordinate ϕ). Because of the axial symmetry, all functions depend on only two spatial variables (η,ξ) and time. This kills a large number of terms involving ϕ derivatives. The nonzero functions in the full four metric are:

α-the lapse function, which is the "time-time" component of the 4-metric, ψ^4-an analytically known function of η and ξ, but time independent, (a, b, c, d)-the four 3-metric functions, the other two zeroed by assuming reflection symmetry.

All five dependent metric variables (α, a, b, c, d) are functions of time and of the two independent spatial variables (η and ξ). We have further simplified the Einstein equations by assuming that the spatial coordinates do not move in the time slices, so that all three shift vector functions are zero (the "time-space" components of the 4-metric). Five terms from the end of the equation in *Fig. 1*, one can find the second time derivative of the 3-metric function $c(\eta, \xi, t)$ term that indicates the hyperbolic nature of the equation.

The equation in *Fig. 1* is the vacuum Einstein equation which states that the Ricci tensor of the 4-metric must vanish. This expression was derived using Macsyma on a SUN workstation and was displayed with Troff output. A further advantage of Macsyma is that one can turn these analytic expressions into error free Fortran expressions which may then be incorporated into the numerical code. Before the help of symbolic computation, many tedious hours were spent in deriving the equations and many tedious weeks were consumed hunting for typographical errors in the code. The process of automating the derivation of equations and the writing of numerical code is at an early stage in its development, but it is likely to be one that will become more important in the future as more complex problems are tackled.

In order to pave the way for the development of a symbolic-numerical interface, NCSA Research Scientist Steven Christensen, himself a quantum gravity researcher, now tends a large garden of algebraic and symbolic computing software (see *Fig. 2*). The symbolic packages which have been most used by relativists are Macsyma, REDUCE, and Maple. With the advent of Mathematica, from Wolfram Research, Inc., Christensen and Leonard Parker, a theorist inter-

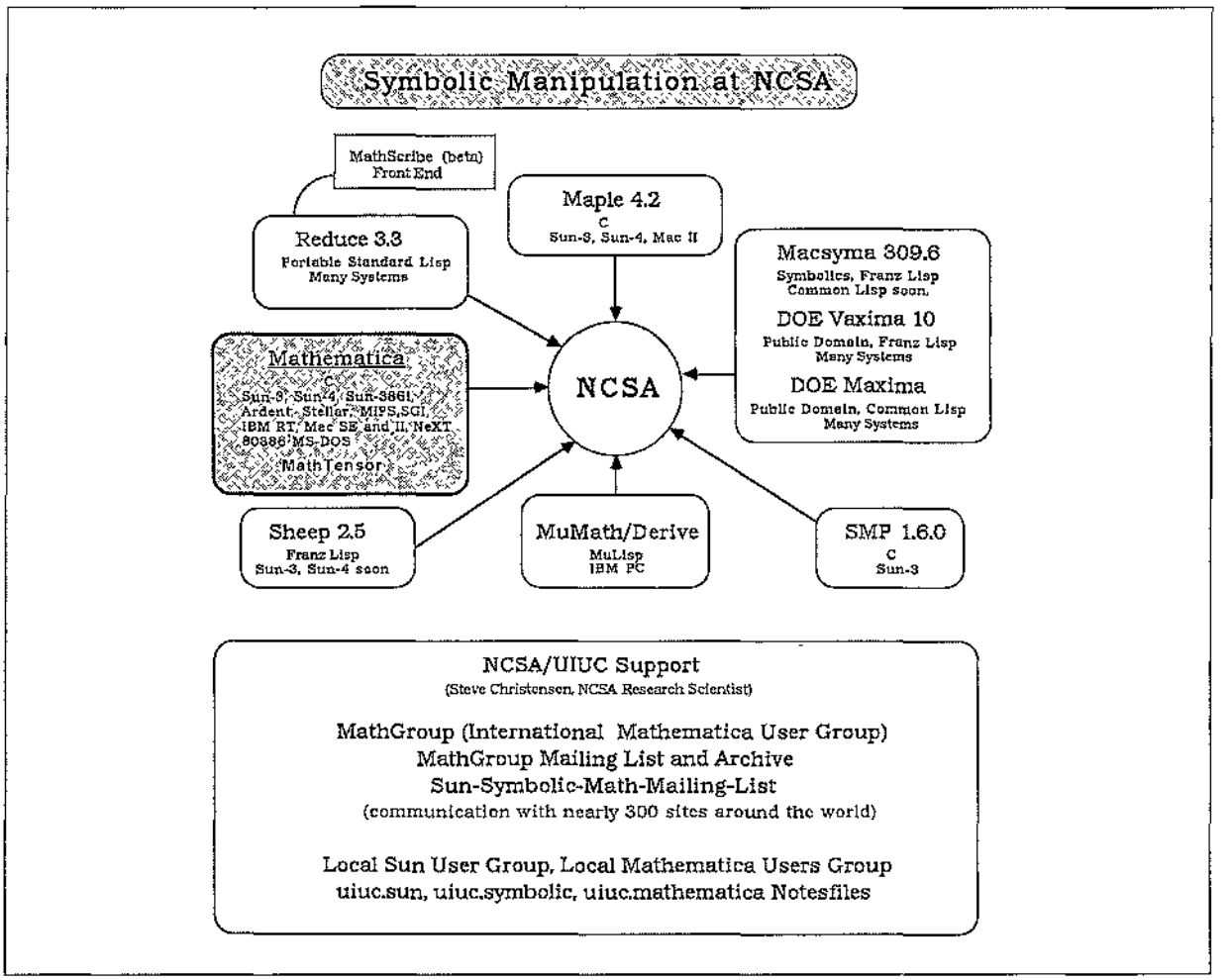


Fig. 2. A schematic diagram illustrating the availability of symbolic computation tools at NCSA (by Steve Christensen).

ested in the cosmology and the quantum nature of gravity at the University of Wisconsin-Milwaukee, are developing an extension to Mathematica that will handle the calculations that need to be performed in general relativity. The integration of these tools into the user interface of the end-users workstation is one of NCSA's goals so that it is as easy to compute symbolically as it is to compute numerically, and so that one can move expressions easily from symbolic into numerical codes.

5. Matter Flows in a Fixed Black Hole Geometry

One of the outstanding computational challenges in numerical relativity is to simulate and understand the behavior of matter in the vicinity of black holes. Most realistic astrophysical black hole systems can be simulated by assuming that the matter interacting with a black hole has very little effect on the black hole itself, i.e. we assume that the mass of the infalling matter is small compared to the mass of the pre-existing hole. In such situations, the dynamics of the matter and fields are of primary importance. This assumption allows the computational problem to be greatly simplified by assuming that the matter moves on a fixed background geometry. Therefore, although one has to solve the fully general relativistic gas dynamics equations, one does not have to solve for the dynamics of the gravitational field as well.

The two major astrophysical environments for accreting black holes are in binary star systems, where the hole would be ten to one hundred times

the mass of our sun, and in quasars or active galactic nuclei, where the central supermassive black hole may range in mass from one million to one billion times the sun's mass. The tendency for gas and stars to concentrate in the central pit of the galactic potential provides a long term source of "fuel" for the central "engines" of these powerful sources of x-rays and radio waves. In binary star systems, the surface of a normal star will be disrupted by the tidal influence of the black hole and the outer material will flow towards the black hole creating a disk of material around it.

Several investigators are using supercomputers to simulate the processes that "feed" the central black holes in galactic nuclei. In very close encounters to the central hole, stars on elliptical orbits may be so tidally deformed on such a short time scale, that nuclear detonation may take place. This process has yet to be simulated completely, but scientists at the Observatoire de Paris (Meudon) have begun to utilize the Cray-1 at the Ecole Polytechnique in attempts to determine if this scenario can actually occur. In a project aimed at determining the capabilities of smooth particle hydrodynamics, Charles Evans and Christopher Kochanek of Caltech have used NCSA's Cray X-MP to successfully simulate the three dimensional tidal disruption of a star as it makes a grazing encounter with a black hole. This will eventually be compared with a three-dimensional finite differenced version now under development.

Assuming such processes can get gas down to the vicinity of the near-hole region, attention then turns to the geometry of the black hole accretion disks, which depends upon the rates of accretion and energy loss. In some cases, the disks may form funnel like gaps above and below the hole through which matter may be accelerated to velocities close to light speed. This has been one explanation for the formation of the large single and double radio jets observed in active galactic nuclei. A complete description of the different geometries and the physical requirements to form them can be found in a review article by Roger Blandford [6].

Jim Wilson, of Lawrence Livermore National Laboratory (LLNL), pioneered the use of supercomputer simulations on the LLNL CDC 6600 to study the flows of material onto black holes [7]. He discovered that a large angle infall will shock as it nears the hole and set up circulation patterns in the matter. Extending his numerical methods a decade later using the Cray-1 at the Max-Planck Institute for Physics and Astrophysics in Munich, Hawley and Smarr [8,9] were able to obtain new and more detailed scenarios for the gas accretion from their simulations. They found that orbiting fat tori around the holes were natural end products of general inflow patterns. These tori could be compared to the simpler analytic predictions of stationary tori by other authors. Later simulations by a variety of researchers studied the 3-dimensional instabilities that such tori may be subject to.

If a large number of charged particles are present in the flow, the gas accreting about a black hole will be ionized and one must take into account not only the motion of the gas, but also electromagnetic interactions. The study of magnetohydrodynamic accretion adds one more dimension of realism to the computer simulations of astrophysical systems containing black holes. Again, Wilson [10] performed the pioneering simulations. More recently, the general analytic equation have been worked out by Sloan and Smarr [11], based upon the formalism of Thorne and MacDonald [12]. Although the introduction of general time-dependent electromagnetic forces in a general relativistic gravitational field pose formidable difficulties, Hawley and Evans [13] recently have written a code that will study properties of realistic axially symmetric magnetohydrodynamic winds interacting with rotating black holes.

Finally, photons interacting with matter become an important contribution when the radiation energy density is of the same order of magnitude as the matter density. This is almost surely the case in the near-hole region of supermassive black holes. The general study of radiation magnetohydrodynamics is one of the Grand Challenges of numerical astrophysics (See Winkler and Norman [14]) and continues to occupy many of the leading researchers in the field. Only within the past year have attempts been made to come to grips with radiation hydrodynamics in black hole physics (see Matzner and Mezzacappa [15] and Bludman & Schinder [16]). These projects are still at an early stage and have been restricted to spherical systems and neutrino rather than electromagnetic fields. It is clear that much progress in (1) the formulation of the physics itself, (2) algorithm development, and (3) large increases in supercomputer speed and memory need to be

made if realistic astrophysical events involving accreting black holes are to be successfully simulated on large scale computing machines.

6. Gravitational Collapse to Form a Black Hole

In the case where the effects of matter become important to the evolution of the black hole, the problems associated with determining the fully dynamic spacetime come into the open. The most studied example of this is the collapse of matter to form a black hole. Although this problem logically comes before the previous one, in which one assumed that a hole was already present, computationally it is much more difficult because one now needs to solve the matter equations fully coupled to the Einstein equations for the gravitational field.

In attempting to understand the process that forms a black hole in the core of a stellar cluster, Stuart Shapiro and Saul Teukolsky at Cornell University have used an IBM 3090V computer to simulate the formation of a spherically symmetric star cluster with 32,000 stars [17]. In their calculation, they follow the orbits of all of the stars as some plunge into the central black hole and as others remain forever in orbit around it. If one wishes to understand the formation of the black hole in the center of a realistic galaxy, then millions of stellar orbits need to be calculated. This may well be necessary in order to predict the detailed from of the 'light spike' that occurs in the galactic center and differentiate between the spike caused by stars clustered around a central black hole and that created by a dense cluster of stars without a black hole.

Most collapse codes, however, are used to model the core collapse of a supernova. A supernovae occurs when the core of a massive star abruptly runs out of fuel for the nuclear reactions taking place within it. The core cools quickly and with the loss of radiation and thermal pressure to support them against gravity, the outer layers of the star collapse, squeezing the core to extremely high density. When nuclear densities are reached the core suddenly becomes very stiff and stops collapsing. The resulting shock wave propagates through the outer layers, causing an explosion which forms the supernova.

The first attempt at numerically modelling this sequence of events was made by May and White in 1966 at LLNL [18]. They modelled a spherically symmetric collapsing star using general relativity. If the mass of the core is small enough, a bounce at nuclear density occurs resulting in a neutron star and an outgoing shock wave. An even greater difficulty lies in the problem of black hole formation during a supernova. In this case the implosion of the outer layers causes the core to become so compressed that it collapses to form a black hole. Because May and White's code did not have the more sophisticated time slicing conditions described below, their code crashed shortly after the hole was formed. Over the years many researchers have repeated their experiments with more accurate physics and more sophisticated numerical methods. This is still an active research field today, particularly in light of the new data coming in from supernovae computations in the context of general relativity.

A spherical collapse, even if general relativistic, cannot emit gravitational radiation. This requires nonsphericity, which up to now has mainly been modeled by assuming axial symmetry. Smarr and Wilson [20] wrote the first such code for nonrotating stellar collapse in 1978 while Smarr was a visitor at LLNL. This was quickly followed with the first successful rotating collapse results by Nakamura and Sato [23]. Since then a number of researchers have developed much more sophisticated codes to handle 2-D (Evans [21] and Stark and Piran [22]) and the beginnings of 3-D (Nakamura [24] and Stark [25]). Questions addressed by these authors include: (1) do rotating matter configurations collapse to form the analytic Kerr rotating black hole solutions?, (2) or can "naked singularities" be formed by collapse?, (3) do such black holes only give off gravitational radiation in the analytically known quasi-normal modes (see for instance [26])?, and (4) does the time honored approximation of using the "quadrupole moment formula" for gravitational radiation hold in a dynamic strong field region of spacetime?

One question often asked is how can one calibrate such complex computer codes. The major tests are against analytic solutions. *Fig. 3* shows a visualization of the gravitational radiation propagation from a "ringing" neutron star which Evans computed at NCSA in 1985. He was able to run his nonlinear code for a perturbed neutron star

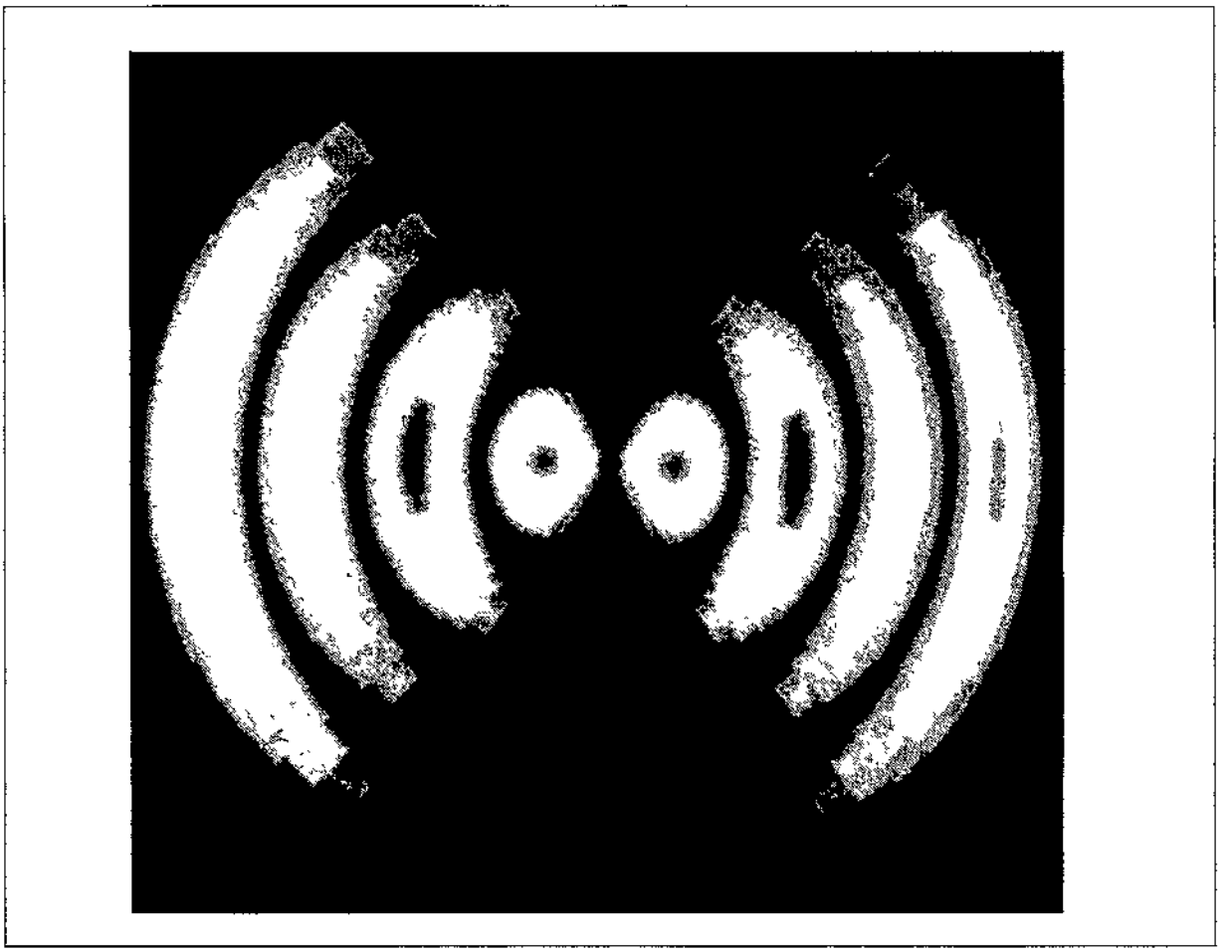


Fig. 3. The gravitational radiation from a fully relativistic neutron star. This image was produced at NCSA by C. Evans.

and compare quantitatively his results with the analytic perturbation theory results of Thorne and his co-workers, which were obtained in the late 1960*s*. The agreement on wave forms was good to about 10% [21]. Since then, new "radiation matching" methods have improved this agreement to a few percent [29].

Our belief is that a great deal of physics remains to be learned in even the axially symmetric regime using accurate and robust codes. Detailed waveforms need to be computed for collapses with realistic equations of state, so that future gravitational wave detectors on earth will know the range of possibilities to expect from stellar core collapse. The exploration of 3-D dynamics is just beginning and will occupy at least the next decade as the more powerful supercomputers and visualization technologies which are required become available.

7. Dynamics of Black Holes

Although one has a dynamical gravitational field in a nonspherical star collapse to form a black hole, the distortions of the hole or the surrounding gravitational field are not that severe. Few realistic examples have found more than a few tenths of one percent of the hole's rest mass is radiated in gravitational radiation. Therefore, one mainly gets to study slightly nonspherical perturbations of the longitudinal and radiative components of the black hole gravitational field.

In more extreme spacetimes involving pure gravitational radiation, such as the interaction of a black hole with a very large amplitude gravitational wave or the collision of multiple black holes, the effects of spacetime curvature can no longer be relegated to a back seat role in the dynamic evolution. While strong gravitational waves and multiple black hole systems may be rare in the physical universe, they nonetheless warrant a careful examination since they provide information on the behaviour of general relativity at its most extreme limits. Since no matter is involved, a study of black hole dynamics deals only with the pure gravitational field and its nonlinear self interaction. It is the ultimate "numerical laboratory" for studying the mathematical physics of nonlinear continuum field theories and their solution space phenomenology.

In a sense, the "purest" spacetimes to study in general relativity are those which contain only gravitational radiation. In the perturbations off of flat spacetime, Einstein showed early on that the waves move at the speed of light and are quadrupolar in lowest angular mode. Brill [27] showed in 1959 that one could solve for initial data that allows for arbitrarily high amplitude waves. Since general relativity is nonlinear, these waves can self-interact and can create a finite mass as measured at infinity. Eppley and Smarr [28] developed the first axially symmetric code at LLNL in the late 1970s to study these finite energy vacuum spacetimes. Since then Stark and Piran [22], Nakamura and Oohara [24], and Abrahams, Evans, and Smarr [29] have developed more accurate codes to compute what happens in highly nonlinear regions. In particular, if the amplitude becomes great enough, such waves can self-focus into a black hole. Note in this case that the "background" space with which to compare it. Such studies are still in the early stages of investigation and we believe will lead to some fundamental insight into the nonlinear dynamics of the pure gravitational field.

At NCSA we have embarked upon a program aimed at developing numerical relativity codes that will be able to calculate the dynamical evolution of black holes in vacuum spacetimes. The goal of our program is to use modern algorithmic, visualization, and distributed computing technologies to create state-of-the-art codes for solving for black hole dynamics. Our program started with writing codes for evolving spherical black holes and refining the numerical techniques to give results accurate to less than 1% [30]. We then developed several 2-D codes which will sequentially study distorted single black holes and then multiple black hole interactions. Our methodology is to calibrate and document each stage of increasing complexity so that a precise tool will emerge for studying the mathematical, physical, and astrophysical questions which involve black hole dynamics.

One of the simplest exact solutions that exists in general relativity is that describing a single non-rotating spherically symmetric black hole and presumably this should be quite easy to reproduce numerically. However as it turned out, the numerical construction of a single black hole with an accuracy that deviated from the analytic solution to within less than one percent was not a trivial problem. Since the solution can be obtained to arbitrary accuracy analytically, this problem provided the perfect test bed for comparing different numerical algorithms useful for numerical relativity. The results of this project was that indeed a new algorithm for solving the evolutions was found that not only produced greater accuracy, but greater stability as well [30].

To avoid evolving the space slices in such a way that they encounter the spacetime singularity at the center of a black hole, thereby crashing the code, we use the so-called "maximal slicing" condition. This imposes an elliptic equation on the lapse function, which has the result that the evolution proceeds a little way inside the hole but then causes the progress of proper time to exponentially fall to zero in the strongest field region. Smarr and York [4] derived an analytic estimate for the *e*-folding time. In *Fig. 4*, one can see that by varying the hyperbolic algorithm (labeled by 1–6) for evolving the finite differenced Einstein equations, one can get the numerical solution for the central value of the lapse function to fall more or less closely to the expected exponential decay (straight line on the semilog graph). The accuracy can also be increased by refining the spatial zone size. *Fig. 4b* shows the results of the six algorithms on a finer spatial mesh than *Fig. 4a*. Note that this decay of the central value of the lapse can be accurately followed to incredibly small values (10^{-50}), even though at the outer radius the value of the lapse function is held to unity. The numeri-

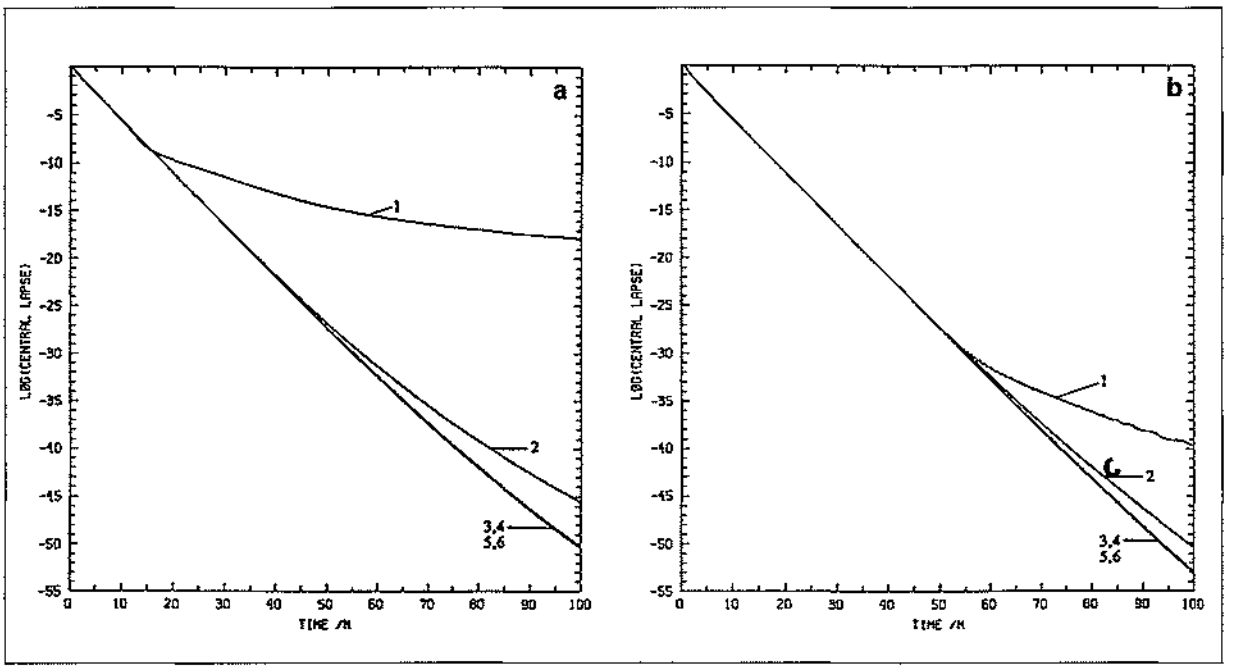


Fig. 4. The logarithm of the central value of the lapse function a in the strong field region of a black hole. The numerical solution approaches the analytic exponential decay when more accurate differencing schemes are use. Reproduced from [30].

cally derived rate of the *e*-folding time agrees with the analytic prediction to three decimal places.

The geometry of such black holes is more complicated than the "deep bowl" form for stellar collapse or gravitational wave collapse to a black hole described above. Here we follow the 1935 model of a "body" in vacuum general relativity proposed by Einstein and Rosen [31]. They suggested that a non-Euclidean geometry consisting of a "throat" connecting two asymptotically flat universes (one of which we live in), represents what we now call a black hole. Thus, the non-Euclidean topology replaces the matter or the gravitational waves as the "source" of mass. This approach allows for a nonsingular geometry both inside and outside of the black hole.

In order to represent the proper curvature of the spacetime produced by the black hole geometry, we embedded a two dimensional subspace (the equatorial plane of the hole) into a flat three-dimensional space to produce the diagram shown in *Fig. 5*. The embedding diagram is computed from the local values of the functions in the 3-metric (ψ^4, *a*, *b*, *c*, *d*) in *Fig. 1*. The two white rings near the reflection point locate the event horizons of the black hole in the upper and lower universe. On this proper space diagram, we can then paint other relevant variables (such as the lapse function in this figure) using a color transfer function. The production of this new method of visualizing simultaneously the curvature in space and in time was the result of an interdisciplinary collaboration at NCSA among scientists Bernstein, Hobill and Smarr, visual artist Donna Cox and computer scientist Ray Idaszak.

In proceeding from a single spherically symmetric black hole to one with less symmetry (but still restricted to axial and reflection symmetry) required an order of magnitude increase in the complexity of the numerical code. In addition to the usual increase in the memory required because one now has a two dimensional array for all variables held in core, the number of variables grows much larger as more terms in the metric tensor become nonzero and their first and second derivatives with respect to two spatial variables must be stored. This increases core requirements for the 2-D code by two orders of magnitude over the 1-D code.

The execution time increases not only because of the number of grid zones, but because of the number of operations that must be performed at

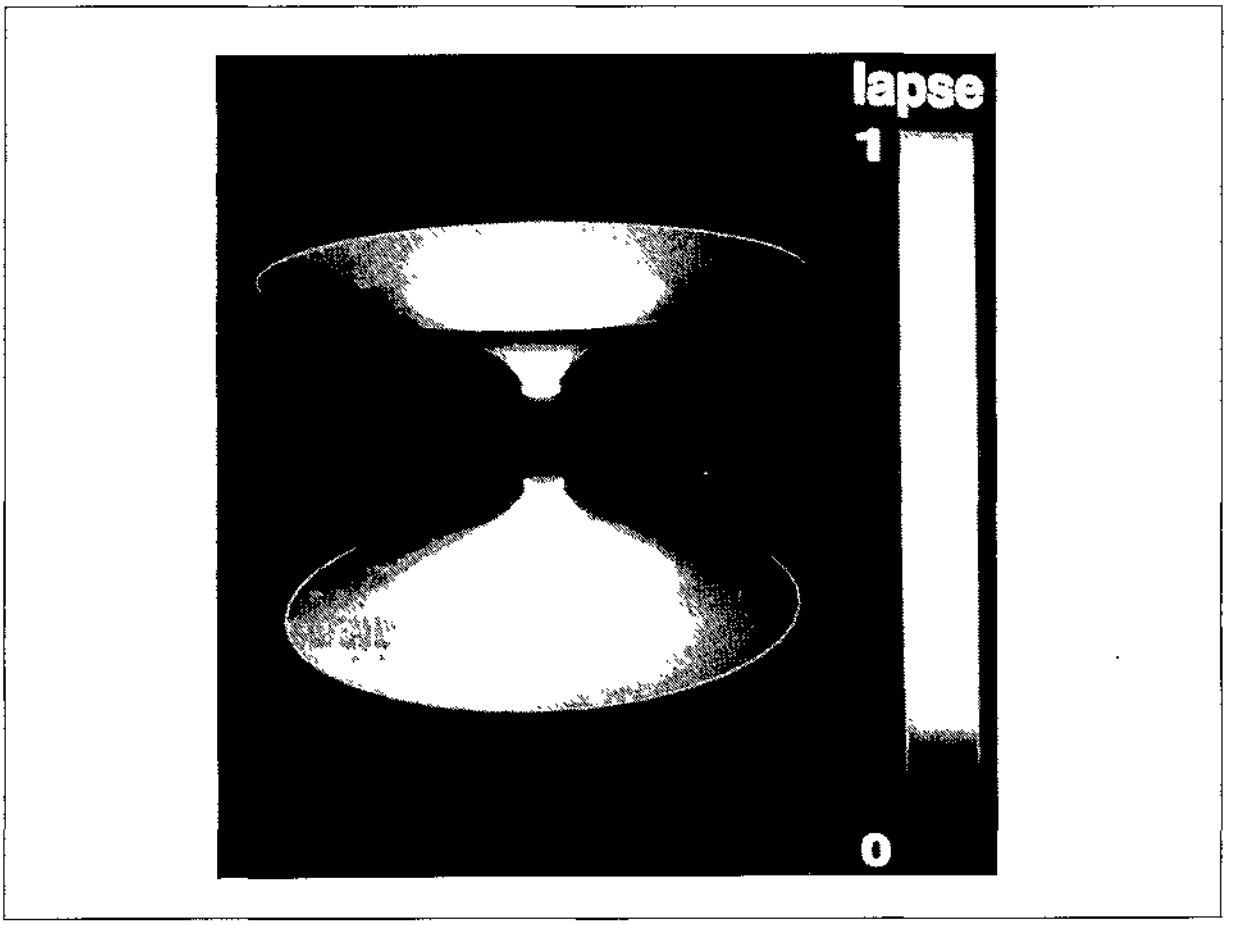


Fig. 5. An isometric embedding diagram of the equatorial plane of a spherically symmetric black hole. The surface is painted with the value of the lapse function as given by the color (here gray scale) transfer function shown at the right. The white circles locate the event horizons of the holes. Color imaging by Cox, Idaszek, and the authors at NCSA.

each zone. First, the length of the field equations grew by a factor of ten. Thus, an equation containing the sum of eight terms in the 1D code, became the sum of eighty terms in the 2D code. Secondly, more hyperbolic evolution equations need to be solved. An equation such as the one for "c" in *Fig. 1* vanishes in the spherical limit. Finally, elliptic equations reduce to ordinary equations in 1D, but they are partial differential equations in 2D, requiring much more sophisticated numerical methods for their accurate solution. At an equal resolution and accuracy (about 10%), the one-dimensional code required five minutes of CPU time on a Cray supercomputer single processor, while the two-dimensional calculation took almost two hours. If an accuracy of less than one percent is the goal, then the projected Cray-2 supercomputer single processor CPU time will be forty hours for a single simulation. Even at a relatively simple stage in its development, numerical relativity demands quite a lot in terms of large scale computing.

Once computed, new techniques are needed to convert the numerical output into recognizable physics. In the past, we have used contour plots or 3-D "height of function" plots to peruse the many variables which define the relativistic gravitational field. For instance, *Fig. 6* shows the gravitational wave variable for two colliding black holes from the work of Smarr and Eppley [32] over a decade ago. Such vector graphics plots can today be run as real time movies on a Macintosh personal computer using NCSA Hypercard animation software [33].

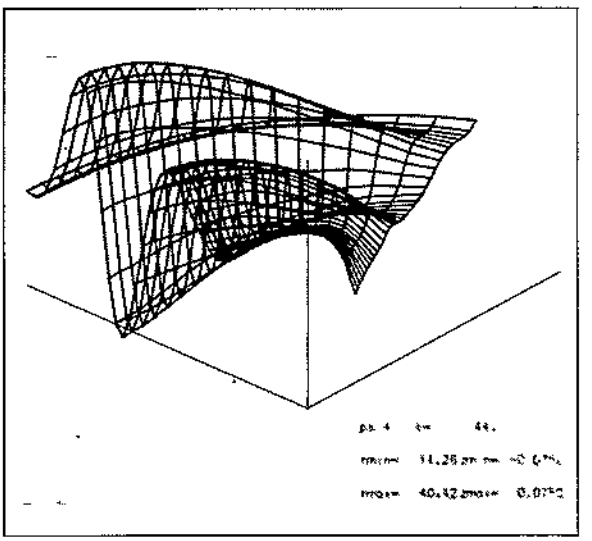


Fig. 6. The gravitational radiation from the head on collision of two black holes which were initially at rest. Computed by Eppley and Smarr in 1978.

Today, new color visualization tools are becoming a major help to the numerical relativist. For instance, using NCSA Datascope, a scientific software tool developed by Tim Krauskopf at NCSA [33], the research scientist can easily view 2-D raster color images of a simulation on workstations ranging from a Macintosh II to a SUN-4. *Fig. 7* is a grey scale image of a large gravitational wave interacting with a black hole made from a Datascope run. One of the great advantages of Datascope is its ability to correlate the floating point numbers, computed by the supercomputer, with their location in a visual image on a windowed workstation. Besides allowing one to quantitatively and qualitatively analyze the physics, it provides one with a powerful tool for locating and diagnosing trouble spots in a numerical simulation. We are also exploring using a combination of the embedding diagram for a single black hole with the gravitational radiation field 'painted' on its surface (instead of the lapse function as in *Fig. 5*) for the case of a distorted black hole which "rings down" to a final spherical state.

The present two dimensional code describing a single vibrating black hole and its production of gravitational radiation is now being tested against the results of linear perturbation theory to check how accurately it is capable of reproducing known analytic results. The extensive analytic results in the literature (see for instance [26]) are computed in a very different coordinate gauge from those used in our numerical simulation. Thus one of the challenges to both numerical and analytic relativists is to come up with new algorithms that yield the gauge transformations between numerical and analytic studies of the same problem. Once the code has been tested in the linear perturbative regime, its real duties will be to calculate the full nonlinear interaction of a strong gravitational radiation with a black hole. Ed Seidel, a specialist in black hole perturbations, will be joining our team in the Fall of 1989 to pursue this comparison and calibration.

If one changes the boundary conditions on the two dimensional code described above, then the problem of two black holes colliding along an axis of symmetry can also be computed. This problem was first attacked by Hahn and Lindquist [34] in the early 1960s, but neither the necessary computing power nor a full understanding of what a black hole is was available at that time. Later with the increased computational power of the CDC-7600 and the hindsight provided by other failed efforts, Smarr and Eppley [32] were finally able to calculate the dynamics of the two black hole collision, albeit with crude resolution and unresolved stability problems. Now that our group has solved both the stability and accuracy problems on the vibrating single black hole problem, our next step is to recompute the two black hole problem to provide unambiguous answers concerning the nonlinear nature of general relativity (for instance, the details of how the two event horizons merge) and gravity wave production in a (rare) astrophysical event.

In addition to the challenge of computing a classical problem in general relativity, the problem of computing and visualizing the geometry of a double black hole spacetime is as yet unsolved. The difficulty rests in creating an embedding diagram of a highly non-symmetrical surface from local information. Unlike the figure of revolution which obtains in the spherically symmetric case of a single black hole (*Fig. 5*), where the embedding surface must be pieced together globally from purely local information contained in the numerically computed 3-metric of the curved space. A decade ago, Eppley and Smarr (see *Fig. 8* taken from [35]) solved this problem using a mechanical process of literally weaving together paper strips

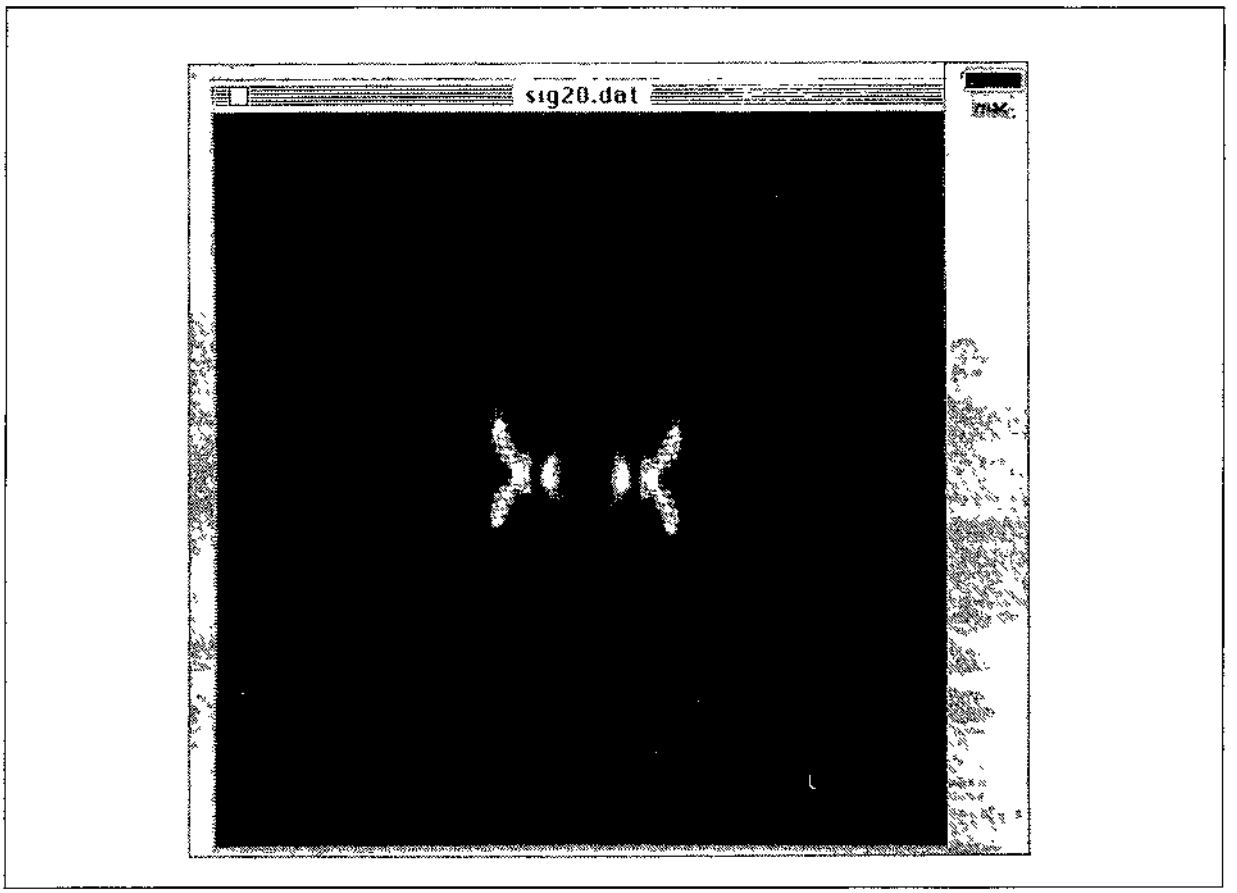


Fig. 7. A measure of the gravitational radiation near a black hole which has been distorted by a large amplitude gravitational wave. The black and white image was shot off the screen of a Macintosh II workstation using NCSA Datascope software.

which contained the individual distorted quadrilaterals produced by each grid zone, each of which could easily be constructed from the local metric information.

A modern computational solution will require collaboration with computational geometers who have long studied embedding algorithms. An example of what may be accomplished today in this field is shown in *Fig. 9* which shows the embedding of a higher dimensional algebraic homotopy in three dimensions. Here the solution of the embedding problem is easier, since one knows analytically the equation which defines the location of 2-surface in 3-space. This image was constructed by mathematician George Francis working once again with the team of Cox and Idaszak.

Finally, the ultimate grand challenge for computational black hole dynamics, is the problem associated with the spiral infall and collision occurring in a black hole binary system. It is expected that this event will produce a significant amount of gravitational radiation; enough to be measured by the next generation of gravity wave detectors. This will require much larger supercomputer memories and speed than available today, if an accurate result is desired. The visualization techniques for imaging volumetric geometry information properly do not exist yet. It is likely to require a group effort among scientists, computer graphics specialists, and computer scientists, with expertise in symbolic computing, numerical algorithms, large code generation, data analysis, and three dimensional visualization software. We believe that the 1990s will see a race between the

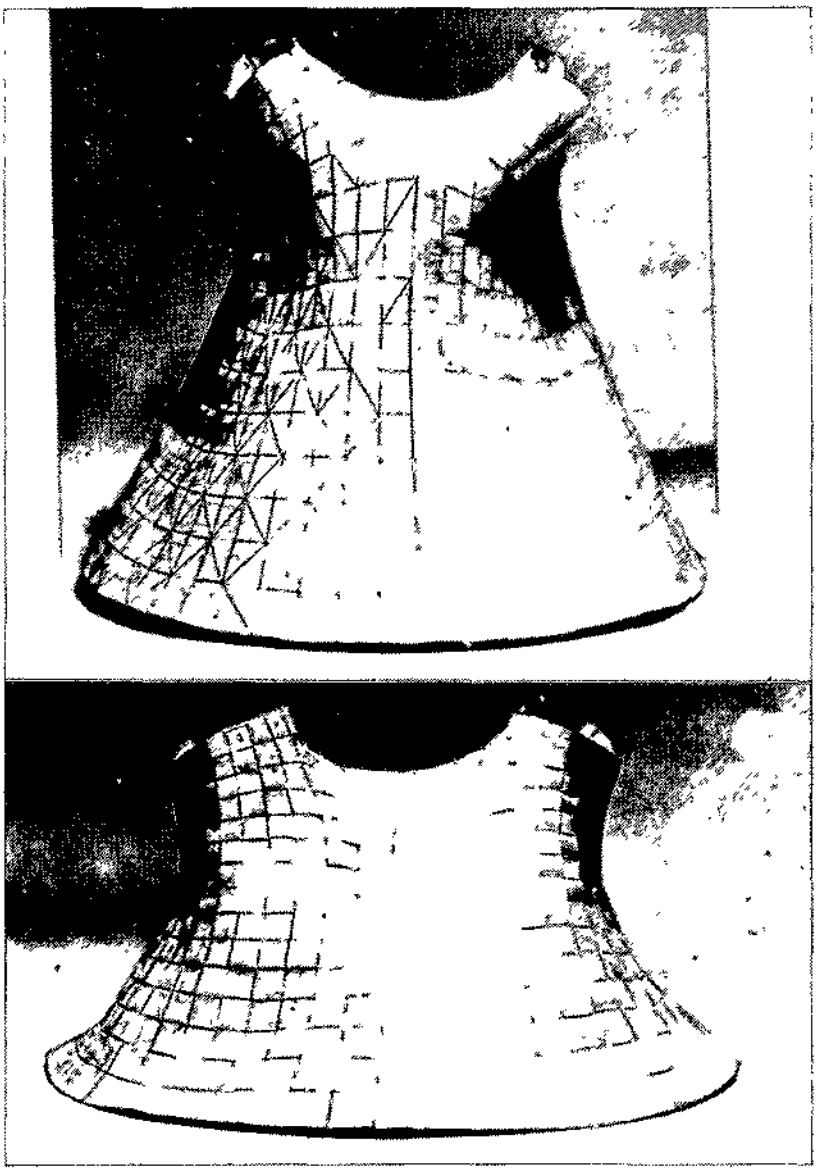


Fig. 8. Paper construction of an isometric embedding diagram of two colliding black holes by Eppley and Smarr in 1978. Note the "double neck" which shows the relative position of the two holes at the moment they begin to move (bottom image) and at a later time in the evolution (top image).

groups attempting to compute such complex spacetimes and those groups attempting to build gravitational radiation detectors sensitive enough to actually measure what the numerical relativists have computed.

8. Conclusions

From the discussion above, it is clear that the understanding of the physics of black holes presents a great computational challenge, for both

theorists and observers alike. Computer aided scientific visualization will become a means for bringing together these two great methodologies of science. With the opening of the Great Observatories in space and on the earth in the 1990s, black holes will move from science fiction to routine astronomy. The computational relativists must continue to advance their field rapidly, if they hope to be partners in this discovery process. A review of how advances in computing technology have influenced numerical relativity can be found in [36].

One major factor limiting computational progress is that unlike the situation that exists in other computational sciences, notably computational chemistry and in some cases computational fluid dynamics, there are no general codes that may be used by the community to explore black hole physics. The number of researchers in the field remains small and each small group has usually written its own code from scratch. If rapid progress is to be made in understanding black hole physics, then the need for a general code becomes all that more acute. This in its own right is a challenge that must be met by numerical relativists.

Of no less importance will be the synergistic relationship that will develop among researchers in mathematics, computer sciences, and the physical sciences. No longer will the domain walls

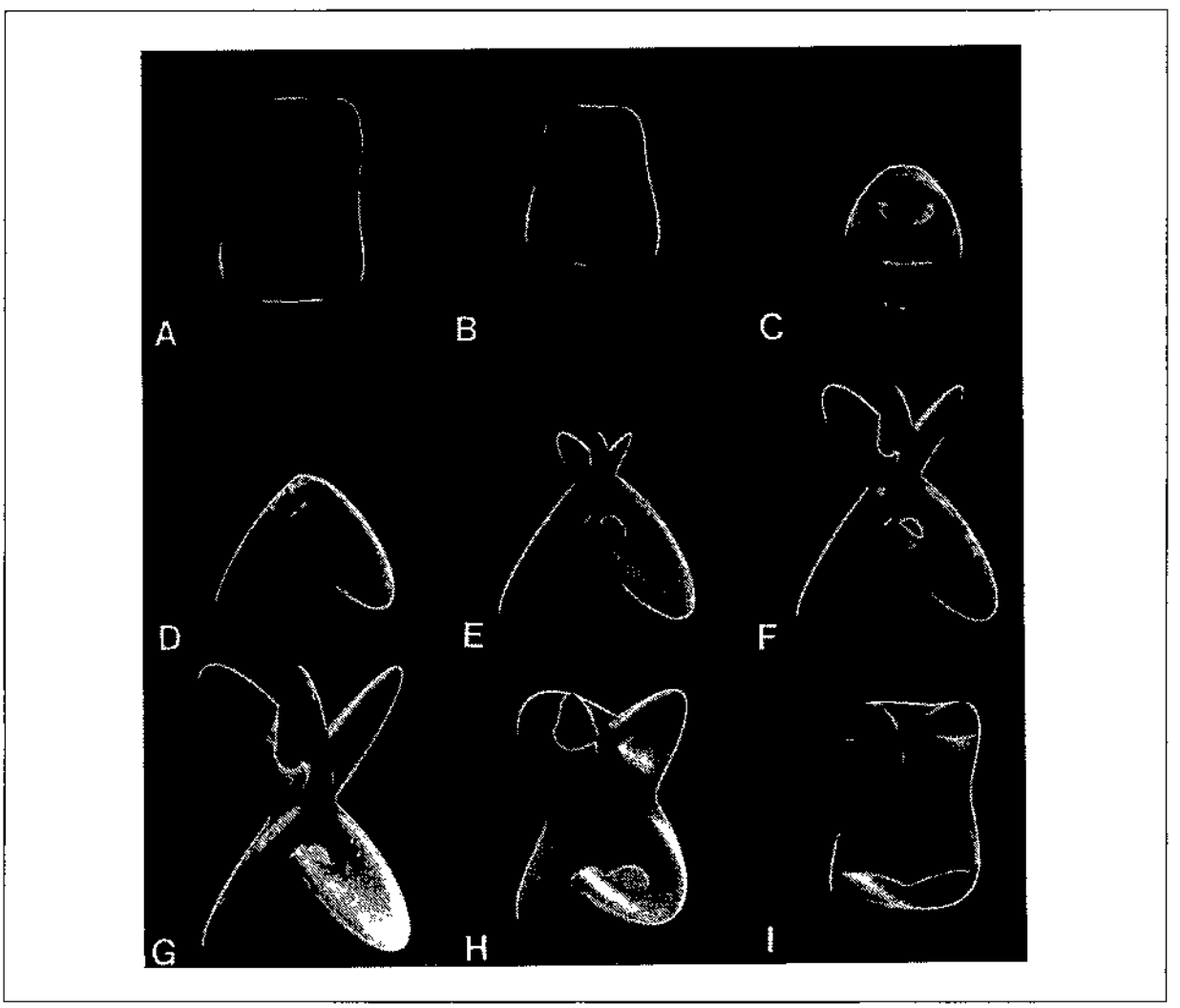


Fig. 9. An embedding of a higher dimensional mathematical object in 3-space, part of the Romboy homotopy, by Cox, Francis, and Idaszek at NCSA.

between these different disciplines be distinct. The techniques used to solve the problems associated with black hole physics require the expertise of people with completely different backgrounds and training. The results obtained in numerical relativity must be accessible to these different subcultures as well.

It is appropriate that this subject be reviewed at the first scientific conference on numerical science held with attendance from the decades old national nuclear weapons laboratories and the recently formed National Science Foundation (NSF) national supercomputer centers. As we noted throughout this paper, the pioneering computations in virtually every aspect of numerical black hole physics came from the nuclear weapons labs. Today, this important branch of theoretical physics and astrophysics is occurring throughout the United States by university researchers having access to the NSF centers. Indeed some of the most ambitious 3D work is being performed in Japan. This rapid spread of the use of advanced computing technology from classified laboratories to American universities to international universities is characteristic of todays vastly changed competitive arena. Those who would be in the vanguard of the science of tomorrow must strive to keep up with the rapidly changing world of computer technology and the emerging fields of computational science.

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