Collision of Two Black Holes

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We study the head-on collision of two equal-mass, nonrotating black holes. We consider various cases, from holes surrounded by a common horizon to holes separated by about $20M$, where M is the mass of each hole. The wave forms and energy output are computed, showing that normal modes of the final black hole are clearly excited. We also estimate analytically the total gravitational radiation emitted, considering tidal heating of horizons and other effects. The analytic calculations, perturbation theory, and strong-field, nonlinear numerical calculations agree very well with each other.

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The collision of two black holes is considered to be one of the most promising and important astrophysical sources of detectable gravitational radiation in our Universe [1]. Since LIGO [1] and VIRGO [1] are expected to begin taking data during this decade, it is important to perform accurate calculations detailing the shape and strength of the wave forms generated during such events. The information gained from the detected wave forms should allow one to reconstruct the astrophysical parameters of the system, and will provide the first direct and unambiguous evidence for the existence of black holes if the unique signature of the quasinormal modes [2] of the hole is excited.

In this paper we present results of numerical studies of time symmetric, axisymmetric head-on collisions of equal-mass black holes. We have been able to extract the wave forms and the total energy emission resulting from the collision. Analysis of the wave forms reveals clearly for the first time that the quasinormal modes of the final black hole are strongly excited from the collision. Although the term quasinormal mode refers specifically to the response of a black hole to infinitesimal perturbations, we also use it here to describe the finite amplitude oscillations of the final black hole. As we will see in the next section, the finite amplitude oscillations are described very well by the true quasinormal modes of the final black hole. This work extends and refines the early work of Smarr and Eppley [3,4] that suggested that the normal modes of the final hole were excited, but the resolution and wave form extraction techniques available at that time did not permit a clean and unambiguous matching to black hole normal modes. The numerical difficulties inherent in this problem also led to fairly large uncertainties in the total energy radiated (Smarr quotes a probable uncertainty factor of 2 [4]). Because of the importance of this fundamental physical problem, we have revisited this calculation with the benefit of more powerful computers and improved analytic and numerical techniques developed over the intervening 15 years to calculate unambiguous wave forms and energy fluxes resulting from the collision.

The initial data sets that we adopt are analytic solutions originally discovered by Misner [5] and subsequently analyzed and evolved by Smarr and Eppley. They are characterized by a parameter μ determining the mass M of a hole and the proper separation L/M of the two holes (see Table I), and consist of two throats connecting identical asymptotically flat spacetimes [6]. In this paper we apply the code described in Ref. [7] to compute evolutions for a family of Misner spacetimes representing equal-mass black holes colliding from distances of between $4M$ and $20M$. If the throats are close enough together (for $\mu < 1.36$) [8] a common apparent horizon surrounds them both, so that the system really represents a single, highly perturbed black hole. For throats separated by more than about $8M$ we are confident that there is no common event horizon surrounding them, as shown by directly integrating light rays (see also a hoop conjecture argument [9]).

Numerically generated wave form templates will be essential for analysis of data collected by future gravitational wave detectors [1]. The method we use to calculate wave forms is based on the gauge invariant extraction technique developed by Abrahams and Evans [10] and applied in Ref. [11] to black hole spacetimes.

For all of the cases studied in this paper we have extracted both the $l=2$ and $l=4$ wave forms at radii of $30M$, $40M$, $50M$, $60M$, and $70M$. By comparing results

TABLE I. The physical parameters of the six initial data sets studied are summarized. M is the mass parameter defined in the text, L/M is the proper distance between the throats, and we note whether or not one apparent horizon surrounds both holes.

μ	M	L/M	Apparent horizon
1.2	1.85	4.46	global
1.8	0.81	6.76	separate
2.2	0.50	8.92	separate
2.7	0.29	12.7	separate
3.0	0.21	15.8	separate
3.25	0.16	19.1	separate

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at each of these radii we are able to check the propagation of waves and the consistency of our energy calculations. In general we find agreement among detectors at different radii to be within 10%–20%. We have also run all the calculations with resolutions of 200×35 (radial, angular) and 300×55 with only minor differences in the $l=2$ wave forms and the total energy radiated. (But as detailed in Ref. [9] the amplitude and precursor of the $l=4$ wave form are rather sensitive to computational parameters.)

In Fig. 1 we show the $l=2$ extracted wave form for the case $\mu=2.2$, for which there is no initial common event horizon. The solid line shows the wave form detected at a distance $r=40M$ and the long-dashed line shows the wave form extracted at $r=60M$. The wave is clearly propagating away from the hole at light speed with essentially invariant shape and amplitude, with a wavelength of $2\times16.8M$, confirming the original findings of Smarr and Eppley [4]. However, our more accurate code now allows us to go beyond estimating the wavelength and to fit quantitatively the wave form to results known from black hole perturbation theory. The short-dashed line shows the result of fitting the $r=40M$ wave form (in the range $64M<t<160M$) to a linear combination of the fundamental and first overtone of the $l=2$ quasinormal mode for the final black hole with mass $2M$. The fit is quite good, matching both the wavelength and damping time, showing that the final black hole mass in indeed very close to the total mass of the spacetime.

In Fig. 2 we show the $l=4$ wave form for the same case, extracted at the same radius. Again this wave form has been fitted (over a similar range) to a superposition of the fundamental and first overtone $l=4$ quasinormal modes of the black hole. Wave forms for all cases studied show similar behavior: The normal modes of the final black hole are excited and account for most of the emitted signal.

We first present a semianalytic determination of the energy radiated by the colliding black holes. Such an estimate is interesting not just as a confirmation of the numerical results, but more importantly, because it provides physical understanding of the numerical data.

We base our understanding of the two black hole collision on the well-studied problem [12,13] of a test point particle m at rest from infinity plunging into a Schwarzschild black hole M. The gravitational wave energy radiated for the test particle case is found to be [12]

$$E=0.0104m^2/M \quad \text{for } m\ll M\,. \tag{1}$$

We build on this accurate result to approximate the energy loss for the case of two unequal-mass black holes of mass m and M, with $m\leq M$. We include correction factors into Eq. (1), so that it can better approximate the two black hole collision. The most important effects one has to take into account [9] are that (i) m may be comparable to M, (ii) the infall is not from infinity, and (iii) the black hole, unlike a point particle, has internal dynamics.

It is useful to first understand the m^2/M dependence of Eq. (1). In Newtonian approximations, the quadrupole moment of the system is $I\sim mr^2$ for $m\ll M$, where r is the radial distance between m and M. The gravitational wave luminosity L is given by

$$L\propto\dddot{I}^2\sim m^2(\dddot{r}\dot{r})^2\sim m^2M^3/r^5\,. \tag{2}$$

As most of the radiated energy is emitted when m is falling near the horizon of M, to obtain the total energy radiated E, we can evaluate the luminosity at $r\sim M$ and multiply it by a time scale of M. This gives $E\propto m^2/M$ as in Eq. (1). Now for m not much smaller than M, it is

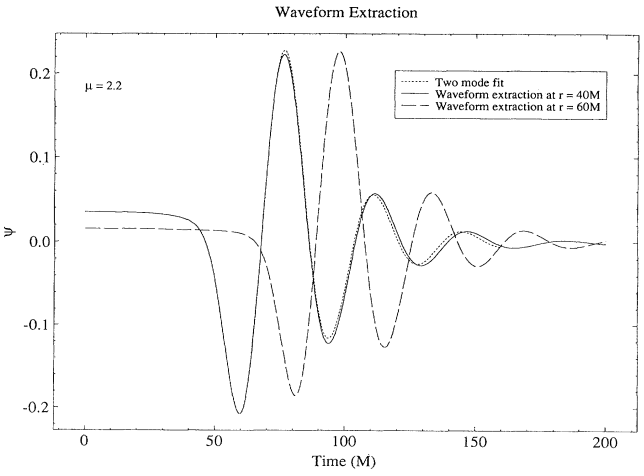


FIG. 1. $l=2$ wave forms for the case $\mu=2.2$. The solid line shows the wave form extracted at $R=40M$ and the long-dashed line shows the wave form at $R=60M$. The short-dashed line shows the quasinormal mode fit.

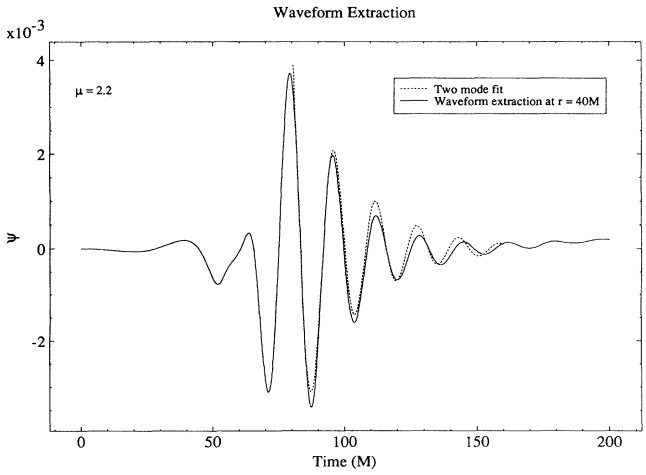


FIG. 2. $l=4$ wave forms for the case $\mu=2.2$. The solid line shows the wave form extracted at $R=40M$ and the short-dashed line shows the quasinormal mode fit.

more accurate to use $I \sim \mu r^2$, with $\mu \equiv mM/(M+m)$. The quadrupole moment formula then leads to [4] $E \propto \mu^2/M$, and we modify Eq. (1) to be

$$E = 0.0104\mu^2/M . \tag{3}$$

Notice that for $m = M$, the μ^2 of Eq. (3) introduces a quite significant factor of $\frac{1}{4}$.

The quadrupole formula (2) also suggests how the expression should be modified when the infall is not from infinity. There are two effects: (i) There is less time to radiate when falling from a finite distance, and (ii) the infalling velocity is smaller for the same separation. The latter effect is much more important. These two effects can be combined into a single factor F_{r_0},

$$F_{r_0} = E_{r_0}/E_\infty = \int_{r_0}^{2M} \dot{r}(\ddot{r})^2 dr \Big/ \int_\infty^{2M} \dot{r}(\ddot{r})^2 dr . \tag{4}$$

Here r is the radial Schwarzschild coordinate, r_0 is the starting point of the infall, and for the infall velocity $\dot{r}$ we use the relativistic expression $(1-2M/r) \times \sqrt{2M/r - 2M/r_0}/\sqrt{1-2M/r_0}$ [14]. We find F_{r_0} ranges from ~ 0.4 (for $r_0 = 6M$) to 1 (for $r_0 = \infty$).

All considerations up to this point are the same whether the infalling object is a point particle or a black hole. As far as the gravitational wave output is concerned, the most important difference between a point mass and a black hole is that a black hole has internal dynamics. There are more channels into which the initial gravitational potential energy in the system can dissipate. Such dissipation decreases the kinetic energy and hence the velocity of the infalling hole, reducing the wave output.

When the hole m is falling in the static gravitational field generated by M, the horizon of m will be deformed by the tidal force. In the membrane paradigm [15] of a black hole, in which the horizon is treated as a 2D surface living in a 3D space, endowed with physical properties such as viscosity, this tidal deformation heats up the horizon. The heating is described by the horizon equations [15,16] $-\dot{\sigma}_{ab} + (g-\theta)\sigma_{ab} + (2\sigma_{ac} + \gamma_{ac}\theta)\sigma^c_b = \epsilon_{ab}$, $-\dot{\theta} + g\theta - \theta^2/2 = \sigma_{ab}\sigma^{ab}$, and $\dot{\gamma}_{ab} = 2\sigma_{ab} + \gamma_{ab}\theta$. Here γ_{ab} is the 2D metric of the horizon, θ and σ_{ab} are the expansion rate and shear of the horizon generators, and $g = 1/4m$ is the surface gravity, while ϵ_{ab} is the normalized electric part of the Weyl tensor ($C_{a\mu b\nu} l^\mu l^\nu$, with l^μ the horizon generators), here to be evaluated on the horizon of the infalling hole m as it falls in the external tidal field M/r^3 of the hole M. For our present purpose, it suffices to solve these equations in the linear approximation, dropping all terms quadratic in the horizon deformation. The Green's function for the linearized equations satisfying the teleological boundary condition is simply $G(t,t') = \exp[g(t-t')]$ for $t < t'$ and $G(t,t') = 0$ for $t > t'$. Hence θ can be obtained by a simple integration. The fraction of the available gravitational potential energy originally in the system which is dissipated into the heating of the horizon of the infalling hole is given by

$$\frac{\Delta m}{m} = \frac{1}{2}\int \theta dt = \frac{1}{2}\int_{r_0}^{2m+2M} [\theta/\dot{r}] dr . \tag{5}$$

The integration in Eq. (5) is terminated at the point when the two holes are engulfed by a common horizon. For $m = M$, the heating on the horizon of M is the same as that on m, so the total fraction of energy going into horizon heating is given by 2 times Eq. (5); i.e., the reduction factor for the energy available for wave generation is $F_h = 1 - 2\Delta m/m$, which decreases with increasing initial separation. For the range of r_0 studied here, F_h gives a reduction from about 3% to 13%.

With these correction factors taken into account, the total gravitational energy radiated is given by

$$E = 0.0104\mu^2/M \times F_{r_0} \times F_h . \tag{6}$$

There are other correction factors in Eq. (6) that we studied and included in Ref. [9]. As their effects are much smaller, we shall not discuss them here. We note that for holes initially infinitely separated, F_{r_0} is 1, while F_h is 0.86. Next we compare Eq. (6) for the total energy radiated to numerical results.

The wave forms shown in Figs. 1 and 2 have been normalized so that the total energy carried away from the system is given by $\dot{E} = \dot{\psi}^2/32\pi$ for all l modes, where ψ is the extracted wave form. The total energy E is computed from this expression. For a check on the accuracy and consistency of this calculation, we have computed E at all five detectors listed above, and at both high and medium grid resolution. We have also computed the energy loss via two independent curvature based indicators, the Newman-Penrose Weyl tensor component Ψ_4 and the Bel-Robinson Poynting vector [4,9]. These constructions yield results consistent with the gauge invariant formulations when computed in the asymptotic far field spacetime [9].

In Fig. 3, we compare results of various analytic estimates of the energy loss to our numerical results for the dominant $l = 2$ wave form calculation. The connected circles show the maximum possible radiation output obtained by comparing the initial black hole masses estimated by the areas of the horizons (or a single horizon if the holes are close enough) to the total mass of the spacetime. For large separation this number approaches 29% as expected from the work of Hawking [17]. The six clusters of unconnected symbols result from our numerical simulations for the six cases studied (listed in Table I). Each symbol corresponds to the $l = 2$ energy computed at each of five "wave detectors" located throughout the radiation zone as before.

The analytic estimate for total gravitational wave energy output given by Eq. (6) is plotted as a dashed line. The agreement of the analytic and the numerical results is highly remarkable, as the analytic results were obtained without prior knowledge of the numerical results, and vice versa. For L/M less than about 9 the analytic formula overestimates the actual energy output computed

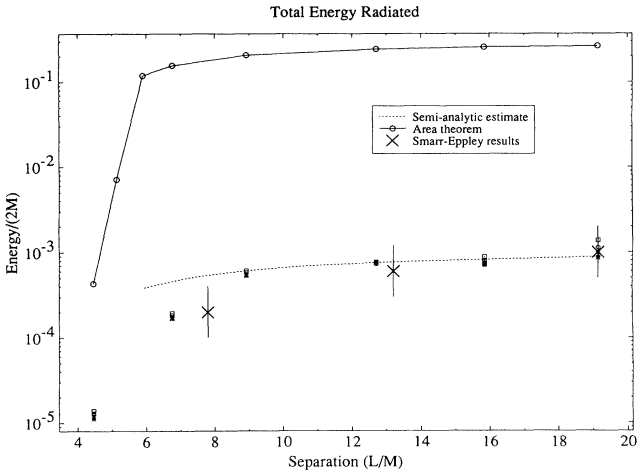


FIG. 3. The total gravitational wave energy output E shown for various cases. All cases shown are resolution (300×55), except the cases $\mu=3.0$ and $\mu=3.25$, which use resolution (200×35). The connected circles are the upper limit based on the area theorem, the clustered symbols show numerical results at various detector locations, the dashed line is the semianalytic estimate, and the crosses show early results by Smarr and Eppley with their approximate error bars.

numerically. This is to be expected since for small enough separation the holes are initially engulfed by a common event horizon and the approximation for colliding black holes is inappropriate. For reference, the early results of Smarr and Eppley are plotted as large crosses with error bars suggested by Smarr [4]. Within the large errors quoted, those early results are remarkably consistent with our results.

We have performed numerical and analytic calculations predicting the gravitational wave forms generated and total gravitational wave energy emitted when two equal-mass black holes collide head on. Both the $l=2$ and $l=4$ wave forms are fitted remarkably well by a superposition of the fundamental and first overtone of the black hole quasinormal modes, and the total energy output is found to be on the order of $0.002M$. The analytic study confirms and elucidates the numerical results, matching the numerical results remarkably well. The total energy output is far below estimates given by a simple application of the area theorem. Taken together, the analytic and numerical results indicate that even for holes that are initially infinitely separated, the total energy output will be the same order of magnitude.

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