THE STRUCTURE OF GENERAL RELATIVITY WITH A NUMERICAL ILLUSTRATION: THE COLLISION OF TWO BLACK HOLES

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This thesis is dedicated to my parents:

Robert L. Smarr, Jr.

Jane Crampton Smarr

No son could wish for more.

THE STRUCTURE OF GENERAL RELATIVITY WITH A NUMERICAL ILLUSTRATION: THE COLLISION OF TWO BLACK HOLES

by

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The splitting of spacetime into space and time is analyzed by abstract methods, coordinate methods, and numerical computer methods. General relativity is assumed to be the correct theory of gravity and the Einstein field equations are studied in their canonical form. A comparison is made of the initial value and evolution equations for gravity in different formalisms (ADM, geometry of spacelike hypersurfaces, timelike congruences). These equations are then illustrated by a number of cosmological and black hole spacetimes. A study of coordinate conditions is undertaken.

The axisymmetric (non-stationary and non-spherical) Einstein equations are discussed, and it is shown how to set up a numerical computer program to integrate these equations starting with a given initial data set. Various applications of this computer approach are discussed: collapse of rotating non-spherical stars, "runaway" collapse, and headon collisions

of black holes. All of these situations would involve generation of gravitational radiation from the formation of black holes.

The particular case of two nonrotating black holes colliding headon is chosen as a test case for the computer. The initial value problem is reviewed and then a computer program is written to generate the spacetime. This generation involves solving four coupled quasilinear hyperbolic (evolution) equations and one linear elliptic equation (maximal slicing) on each spacelike sheet. The numerical difficulties are discussed and graphical methods are used to present the result of the computations.

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Chapter I

The subject of gravitational collapse has been studied since the pioneering work of Oppenheimer and Snyder in 1939. It received new theoretical impetus in the early 1960's from Wheeler and his coworkers. The discovery of pulsars and quasars, Weber's reported gravitational wave pulses, and the continued study of supernovae sustained this interest in the physics of strong gravitational fields throughout the sixties. Now with the probable discovery of a black hole in Cygnus X-1, the need for realistic models of collapse phenomena is crucial.

In this thesis I shall outline the machinery needed to probe the structure of general relativity by numerical studies of such collapse models. Our approach will be that of slicing spacetime into spacelike hypersurfaces. Rather than analyze a single calculation, we shall attempt to build up a formalism for comparing the various types of calculations which have been made and for pointing the way to new attacks on unsolved problems.

In Chapter II we describe the abstract structure of spacetime as seen from a space + time split. Given a stack of spacelike hypersurfaces we set up the projection calculus which allows us to keep four dimensional covariance while essentially dealing with objects on the 3-surfaces. This leads to two splittings of the Riemann tensor which point out the

difference between various "physical components" of the curvature. We see that fourteen of the twenty components describe how the 3-slice is embedded in spacetime and that six of the components are dynamic. The former lead to the constraint equations while the latter yield the evolution equations. The field variables are taken to be the intrinsic metric and extrinsic curvature tensors. The time derivative is geometrically the Lie derivative, and we illustrate the abstract nature of the lapse function and shift vector of ADM. Finally, we obtain some new expressions for the invariants of the spacetime Riemann tensor.

In Chapter III we study the congruences which are the dual notion to hypersurfaces. There are three natural congruences on the spacetime which arise in general: the congruence normal to the hypersurfaces, the coordinate line congruence, and the congruence of matter flows if matter is present. An analysis of the acceleration, expansion, shear, and twist of an arbitrary congruence is made. We derive propagation equations for these hydrodynamical quantities using the Lie derivative along the congruence instead of the covariant derivative along the congruence as is usually done. Then we show how the ADM coordinate system naturally arises, and we rederive their equations by simply taking the spatial part of the equations derived in Chapter II.

The initial value problem is set up and discussed in Chapter IV. We review the history and current state of

the conformal method for solving the constraints. We point out what the freely specifiable parts of the initial data are and discuss the physical significance of each item in the theory. A brief review is made of the special cases of initial data which have been investigated in the past.

The evolution of this data is the subject of Chapter V. We describe some of the global aspects of the Cauchy problem and indicate how this is related to coordinate conditions on the lapse function. We look in detail at the geodesic normal and maximal slicing as examples of such conditions. We point out how the shift vector may be used to recoordinatize the surfaces. Then the main physical aspects of time dependent spacetimes which we hope to explore in our slicings are reviewed. These include singularities, horizons, and gravitational radiation. It is briefly indicated how these three phenomena may be observed on the computer.

Chapter VI applies the above abstract tools to the study of some well known spacetimes. We compare in detail various slicings of the Schwarzschild or single nonrotating black hole spacetime. The Kerr black hole and spatially homogeneous cosmologies are used as examples of spacetimes with several congruences present. These illustrate the use of a shift vector and noncomoving coordinates respectively. Finally, previous collapse calculations are described in terms of our framework of hypersurfaces and congruences.

The last two chapters VII and VIII deal with an

ongoing research problem: the collision of two black holes. This spacetime contains most of the features of a general time dependent solution of Einstein's equations. We first review the knowledge of the time symmetric initial data. Then we map out the general features of the horizon, radiation, and singularity we hope to monitor. Several estimates of the energy loss through radiation are made. We then review the astrophysical setting in which such processes might occur.

In Chapter VIII we show how to set up the evolution equations on the computer. Our general approach is to make a formalism general enough so that with little alteration it could be used to study other axially symmetric problems, such as the nonspherical collapse of stars. The hyperbolic and elliptic equations which arise are discussed, and alternate methods of coding them on a two dimensional grid are examined. Finally, preliminary numerical results on the collision are given with graphs showing the evolution of certain key geometric quantities. No final results on radiation or horizon formation have yet been achieved.

Chapter II

Before we study particular coordinate systems, we should strive to ascertain what is the geometric nature of the problem.[1] Therefore, a summary of the coordinate free approach to spacelike slicings of spacetime will be presented first.

We may define a slicing of spacetime by choosing a scalar function t with timelike gradient $\nabla_a t$. Then the slices S_t are given by t = constant with future pointing unit normal vector n_a given by:

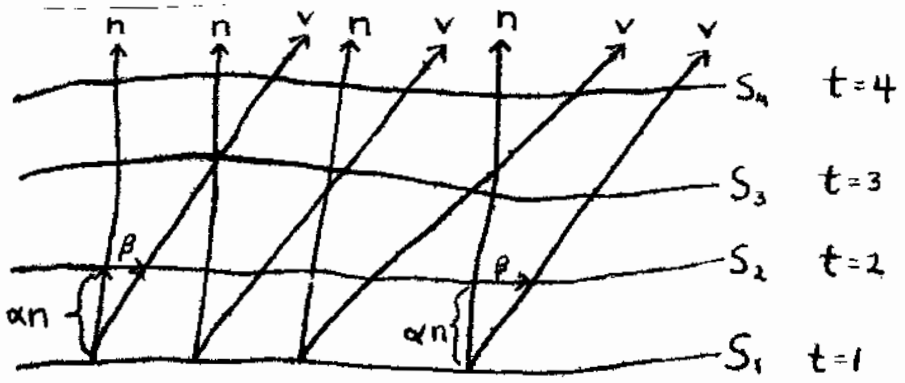
$$n_a = -\alpha \nabla_a t \qquad n_a n^a = -1 \qquad \alpha = [-(\nabla_a t)(\nabla^a t)]^{-\frac{1}{2}}$$

Let the spacing of the slices be governed by the lapse function α. That is, the proper time along the unit normal from slice S_t to slice S_{t+dt} is $d\tau = \alpha dt$.

Choose a spatial coordinate grid on some initial slice. We will propagate it from slice to slice along some timelike congruence v. In other words, along any member of the congruence we have x_i = constant (i = 1,2,3). The shift vector β_a is defined as the difference, in the hypersurface S_t, between the normal connecting vector αn_a and the tangent vector v_a to the coordinate line congruence:[2]

$$v_a = \alpha n_a + \beta_a \qquad \text{with } n_a \beta^a = 0$$

We may picture these spacetime vector fields as follows:



The choice of α and β_a is arbitrary, subject to maintaining $v_a v^a < 0$. This is a consequence of the general covariance of general relativity. The physics is independent of which sequence of slices we choose ("the many fingered nature of time"—reflected by α) or of the coordinates we lay down in each slice (reflected by β_a).

Therefore, starting from an initial slice we can sweep out the same piece of spacetime with many different sets of vector fields (n_a, β_a, v_a) and slices S_t. This means there are two types[3] of 4-tensors on the spacetime: slicing independent tensors like g_{ab}, R_{ab}, R_{abcd}, ε_{abcd}, ∇_a, T_{ab}, etc. and slicing dependent tensors such as h_{ab}, $\mathcal{R}_{ab}$, $\mathcal{R}_{abcd}$, ε_{abc}, D_a, S_{ab}, etc. (see below for definitions). Of this last set there is a subclass of <u>spatial tensors</u> A_{ab} where $A_{ab}n^a = A_{ab}n^b = 0$.

Two spatial tensors of particular interest are the <u>intrinsic metric tensor</u> or first fundamental form h_{ab}:

$$h_{ab} = g_{ab} + n_a n_b$$

and the extrinsic curvature tensor or second fundamental form K_{ab}:

$$K_{ab} = -h_a{}^c h_b{}^d \nabla_c n_d$$

This pair (h_{ab}, K_{ab}) will be regarded as the dynamical variables in what follows. We will see that they must satisfy certain constraint equations on a given slice and we shall find the evolution equations carrying them from one slice to another along the v congruence.

For both of these problems we will need derivative operators, based on the spacetime covariant derivative ∇_a, but adapted to the slicing congruence under study. For spatial tensors $T^a{}_b$, the spatial covariant derivative $D_c T^a{}_b$ of $T^a{}_b$ is also a spatial tensor if:

$$D_c T^a{}_b = h_c{}^m h^a{}_n h_b{}^p \nabla_m T^n{}_p$$

The time derivative is generalized by the notion of the Lie derivative[4] along a congruence:

$$\pounds_v T^a{}_b = V^c \nabla_c T^a{}_b - T^c{}_b \nabla_c V^a + T^a{}_c \nabla_b V^c$$

We will shortly introduce the ADM coordinate system in which

we will find that the above objects reduce to the usual coordinate objects more familiar to us.

Just as in electrodynamics it is useful to break up the Maxwell tensor F_{ab} into the electric field $E_a = n^b F_{ba}$ and the magnetic field $B_a = \frac{1}{2}\varepsilon_{abcd}F^{bc}n^d = \frac{1}{2}\varepsilon_{abc}F^{bc}$ on the hypersurface S_t (with normal n_a), here also it is useful to break up the Riemann curvature tensor R_{abcd} into its "physical components"[5] on a hypersurface S_t. There are two equivalent ways to proceed here.

The first split is essentially a geometric one. From classical differential geometry one knows that there are formulae relating the curvature of a subspace to the curvatures of a space of one higher dimension in which it is embedded (equations of Gauss and Codazzi).[6] These are most easily seen if one splits[7] the Riemann tensor of the spacetime into its fully projected part P_{abcd}:

$$P_{abcd} = h_a{}^m h_b{}^n h_c{}^p h_d{}^r R_{mnpr} \quad ,$$

a part with one normal A_{cab}:

$$A_{cab} = h_a{}^m h_b{}^n h_c{}^p n^r R_{mnpr} \quad ,$$

and a part with two normals Q_{ab}:

$$Q_{ab} = n^m h_a{}^n h_b{}^p n^r R_{mnpr} \quad .$$

We will see that P_{abcd} and A_{cab} are totally determined on S_t if we are given (h_{ab}, K_{ab}) on S_t. On the other hand, Q_{ab} represents the six dynamical components of the curvature tensor R_{abcd}, and requires field equations (such as $R_{ab} = 0$) as well as (h_{ab}, K_{ab}) if it is to be evaluated on S_t.

The other split is based on the algebraic decomposition of R_{abcd} into the Weyl conformal tensor C_{abcd} and the Ricci tensor $R_{ab} = R^{c}{}_{acb}$. We can then regard the Weyl tensor as representing the gravitational field while the matter variables give us R_{ab} through the Einstein field equations:

$$R_{ab} - \tfrac{1}{2}g_{ab}R = T_{ab} \quad .$$

The natural spatial tensors E_{ab}, B_{ab}, ρ, t_a, S_{ab} then arise as:

$$E_{ab} = n^{c}n^{d}C_{cabd} \qquad B_{ab} = \tfrac{1}{2}\varepsilon^{cd}{}_{a}n^{m}h^{n}{}_{b}C_{mncd}$$

$$\rho = n^{a}n^{b}T_{ab} \qquad t_a = n^{d}h_{a}{}^{b}T_{bd} \qquad S_{ab} = h_{a}{}^{c}h_{b}{}^{d}T_{cd} \quad .$$

We will call E_{ab} the "electric" and B_{ab} the "magnetic" part of the Weyl tensor,[8] while ρ will be called the mass-energy density, t^a the momentum density, and S_{ab} the stress density of the matter fields.

These two splittings together with the equations allowing us to relate them can compactly be written as:

$$R_{abcd} = 4n_{[a}Q_{b][c}n_{d]} + 2n_{[a}A_{b]cd} + 2n_{[c}A_{d]ab} + P_{abcd}$$

$$C_{abcd} = 4n_{[a}E_{b][c}n_{d]} + 2n_{[a}B_{b]}{}^{m}\varepsilon_{mcd} + 2n_{[c}B_{d]}{}^{m}\varepsilon_{mab} + \varepsilon_{abm}E^{mn}\varepsilon_{ncd}$$

$$T_{ab} = \rho n_a n_b - 2n_{(a}t_{b)} + S_{ab}$$

$$R_{abcd} = C_{abcd} - g_{a[d}R_{c]b} - g_{b[c}R_{d]a} - \tfrac{1}{3}Rg_{a[c}g_{d]b}$$

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = T_{ab}$$

$$R_{ab} = Q_{ab} - P_{ab} - Qn_a n_b - 2n_{(a}t_{b)}$$

$$R = -P + 2Q$$

It is convenient to introduce two more objects M_{ab} and P_{ab} defined by:

$$M_{ab} = \tfrac{1}{2}A_{amn}\varepsilon^{mn}{}_{b} \qquad P_{ab} = -P^{c}{}_{acb} \qquad P = P^{a}{}_{a}$$

so we can write:

$$A_{cab} = M_{c}{}^{e}\varepsilon_{eab} \quad \text{and} \quad P_{abcd} = \varepsilon_{abm}(P^{mn} - \tfrac{1}{2}h^{mn}P)\varepsilon_{ncd} \ .$$

In this way it is apparent that there are two subsets of each split which transform among themselves: "electric type" quantities

$$(E_{ab}, \rho, S_{ab}) \leftrightarrow (P_{ab}, Q_{ab})$$

and "magnetic type" quantities

$$(B_{ab}, t_a) \leftrightarrow (M_{ab}) \quad .$$

The explicit relations are derived in the Appendix:

$$S_{ab} = Q_{ab} - P_{ab} - \tfrac{1}{2}(2Q - P)h_{ab}$$

$$S = -\tfrac{1}{2}(4Q - P) \qquad t_a = h^{cb}M_c{}^e\varepsilon_{eba}$$

$$\rho = -\tfrac{1}{2}P \qquad B_{ab} = M_{(ab)}$$

$$E_{ab} = \tfrac{1}{2}(Q_{ab} + P_{ab}) - \tfrac{1}{6}(Q + P)h_{ab}$$

$$Q_{ab} = E_{ab} + \tfrac{1}{2}S_{ab} - \tfrac{1}{6}(\rho + 2S)h_{ab}$$

$$P_{ab} = E_{ab} - \tfrac{1}{2}S_{ab} - \tfrac{1}{6}(4\rho + S)h_{ab} \qquad \{M_{ab} = B_{ab} + \tfrac{1}{2}\varepsilon_{acb}t^c\}$$

$$Q = -\tfrac{1}{2}(\rho + S)$$

$$P = -2\rho$$

We note that there are 12 components of R_{abcd} in the "electric" subset and 8 in the "magnetic" subset. For the first splitting we have:

P_{ab}	6 components	$P_{ab} = P_{(ab)}$	
Q_{ab}	6 components	$Q_{ab} = Q_{(ab)}$	
M_{ab}	8 components	$M_{(ab)} = B_{ab}$	$M_{[ab]} = \tfrac{1}{2}\varepsilon_{acb}t^c$

and for the second splitting we have the usual 10-10 split between the Weyl and Ricci tensors:

E_{ab}	5 components	$E_{ab} = E_{(ab)}$	$E^a{}_a = 0$
B_{ab}	5 components	$B_{ab} = B_{(ab)}$	$B^a{}_a = 0$
ρ	1 component		
t_a	3 components	$n^a t_a = 0$	
S_{ab}	6 components	$S_{ab} = S_{(ab)}$	

In a vacuum spacetime ($T_{ab} = 0$), the Ricci tensor vanishes:

$$R_{ab} = 0 \Longleftrightarrow \rho = t_a = S_{ab} = 0 \quad ,$$

and the Weyl tensor is identical to the Riemann tensor:

$$C_{abcd} = R_{abcd} \Longrightarrow Q_{ab} = E_{ab} = P_{ab} \qquad M_{ab} = B_{ab}$$

We now wish to derive several important relations between these components of the spacetime Riemann tensor, the <u>intrinsic curvature tensor</u> $\mathcal{R}_{ab}$, the extrinsic curvature tensor K_{ab}, and the <u>intrinsic Riemann tensor</u> $\mathcal{R}_{abcd}$ on a given S_t. We will verify our remark that Q_{ab} contains the only dynamical components of R_{abcd} and obtain the four constraint equations on (h_{ab}, K_{ab}).

To proceed we need to establish a few lemmas. The proofs of these statements and later theorems are contained in the Appendix. The <u>proper acceleration</u> a_b of the timelike curve with n_b as its unit tangent vector is given by:

$$a_b = n^a \nabla_a n_b \quad .$$

It is given in terms of α by:

$$a_b = D_b \ell n \alpha \quad .$$

Using this, we can show from the definition of K_{ab} that:

$$K_{ab} = -\nabla_a n_b - n_a a_b = -\nabla_a n_b - n_a D_b \ell n \alpha \quad .$$

Repeated use of the last equation together with the definitions of the spacetime and space Riemann tensors:

$$R_{abcd}\, K^d = 2\nabla_{[a}\nabla_{b]} K_c$$

$$R_{abcd}\, \ell^d = 2D_{[a}D_{b]}\ell^d \qquad n^a \ell_a = 0$$

yields P_{abcd} (the equation of Gauss):

$$h_a{}^m h_b{}^n h_c{}^p h_d{}^r R_{mnpr} = P_{abcd} = R_{abcd} + 2K_{c[a}K_{b]d}$$

$$- P_{ab} = R_{ab} + KK_{ab} - K_{ca}K^c{}_b$$

$$- P = R + K^2 - K_{ab}K^{ab}$$

Notice that P_{abcd} is completely determined by (h_{ab}, K_{ab}) on S_t; it involves no time derivatives of K_{ab}. Thus, it is a

constraint equation between (h_{ab}, K_{ab}). Using the field equations for P from above, we have the Hamiltonian constraint equation:

$$-P = R + K^2 - K_{ab}K^{ab} = 2\rho = 2T_{ab}n^a n^b$$

Similarly, one can find A_{cab} (the equation of Codazzi) and, with the field equations, t_b:

$$A_{cab} = -2D_{[a}K_{b]c} = h_a{}^m h_b{}^n h_c{}^p n^r R_{mnpr}$$

$$t_b = D_b K - D_a K^a{}_b = h_b{}^c n^a T_{ac}$$

the latter being the momentum constraint equation.

This leaves only the six components of Q_{ab} undetermined. We will show these are dynamical by deriving the evolution equations for (h_{ab}, K_{ab}) and noting that only Q_{ab} appears in them.

To provide insight into the structure of the evolution equations[9] as well as into the nature of the coordinate freedom embodied in α and β_a, we will start with the vacuum equations using geodesic normal coordinates ($\alpha = 1$, $\beta_a = T_{ab} = 0$), then add matter, a lapse function, and finally a shift vector. The derivations of these equations are contained in the Appendix.

As mentioned above, the Lie derivative is the

geometric version of the partial time derivative ∂_t familiar from coordinate versions of Einstein's equations (e.g. ADM below). With geodesic normal coordinates $\partial_t \leftrightarrow \pounds_n$. Since the basic field variables of a space + time split are (h_{ab}, K_{ab}) we look for $\pounds_n h_{ab}$ and $\pounds_n K_{ab}$. The result is:

$$\pounds_n h_{ab} = -2K_{ab}$$

$$(\alpha = 1, \beta_a = 0)$$

$$\pounds_n K_{ab} = -K_{ac}K^{c}{}_{b} - Q_{ab}$$

Here we see the dynamical character of Q_{ab}. These are the same "tidal acceleration" terms of the Riemann tensor which drive the geodesic deviation equation[10] (or a Weber-type gravitational wave antenna). Indeed, we will see later that K_{ab} gives the expansion and shear of the n congruence.

To make the system of equations self-contained we need to express Q_{ab} in terms of h_{ab}, K_{ab}, ρ, and S_{ab}. When the field equations hold one can use our previously derived results relating Q_{ab} to P_{ab} and S_{ab} plus the Gauss equation relating P_{ab} to $\mathcal{R}_{ab}$ and K_{ab} to give:

$$Q_{ab} = -\mathcal{R}_{ab} - KK_{ab} + K_{ca}K_{b}{}^{c} + S_{ab} - \tfrac{1}{2}Th_{ab}$$

$$\text{where } T = -R = g^{ab}T_{ab} = -\rho + S$$

Specializing to the vacuum one can write the evolution

equations as:

$$\pounds_n h_{ab} = -2K_{ab}$$

$$(\alpha = 1,\ \beta_a = T_{ab} = 0)$$

$$\pounds_n K_{ab} = -2K_{ca}K_b{}^c + KK_{ab} + \mathcal{R}_{ab}$$

The structure of these equations is not only simple but physically quite suggestive. There are two types of terms driving the evolution: a "curvature potential" term $\mathcal{R}_{ab}$ built out of only spatial derivatives of h_{ab} and a "kinetic energy" term $KK_{ab} - 2K_{ca}K_b{}^c$[11] quadratic in the velocities or first time derivatives of h_{ab}. It is surprising that the spatial derivatives segregate themselves completely from the time derivatives in such a complicated partial differential system.

When matter is present, the only change is that the stress S_{ab} and the mass-energy density ρ also drive the evolution through Q_{ab}.

$$\pounds_n h_{ab} = -2K_{ab}$$

$$(\alpha = 1,\ \beta_a = 0)$$

$$\pounds_n K_{ab} = -2K_{ca}K_b{}^c + KK_{ab} + \mathcal{R}_{ab} - S_{ab} + \tfrac{1}{2}Th_{ab}$$

When we drop the geodesic requirement ($\alpha = 1$), we develop a different congruence and slicing of the spacetime. There are two main physical results. First, t is no longer

proper time along the congruence. That is, varying α over S_t varies the rate at which "clocks tick" along any given member of the αn congruence. Second, the acceleration of the congruence is in general nonzero and this will presumably show up as a new driving term in the evolution equations.

Because the addition of the nontrivial lapse function actually gives a new slicing as well as a new congruence αn, there is no simple prescription for inserting the proper α's in our previous evolution equation. Also, the time derivative has now become $\pounds_{\alpha n}$. Another derivation from the Appendix yields:

$$\pounds_{\alpha n} h_{ab} = -2\alpha K_{ab}$$

$$(\beta_a = 0)$$

$$\pounds_{\alpha n} K_{ab} = -\alpha K_{ac} K^{c}{}_{b} - D_a D_b \alpha - \alpha Q_{ab}$$

and when the field equations hold:

$$\pounds_{\alpha n} K_{ab} = -2\alpha K_{ac} K^{c}{}_{b} + \alpha K K_{ab} + \alpha R_{ab} - D_a D_b \alpha - \alpha S_{ab} + \tfrac{1}{2}\alpha T h_{ab}$$

As expected, an acceleration term appears:

$$-D_a D_b \alpha = \alpha [D_a a_b + a_a a_b] \quad .$$

Finally, we may wish to use a nonnormal coordinate system. Since we can regard this as keeping the slicing

given by αn but changing the coordinate line congruence to $v = \alpha n + \beta$, it is not too surprising that the only effect on the evolution equations is to replace[12]

$$\pounds_{\alpha n} \to \pounds_v - \pounds_\beta$$

where $\partial_t \leftrightarrow \pounds_v$ (i.e. along v the space coordinates are fixed while t varies). Recalling that S_{ab}, h_{ab}, etc. still refer to the n congruence, we can obtain the evolution equations in their full generality as:

$$(\pounds_v - \pounds_\beta)h_{ab} = -2\alpha K_{ab}$$

(general)

$$(\pounds_v - \pounds_\beta)K_{ab} = -2\alpha K_{ca}K_b{}^c + \alpha K K_{ab} + \alpha R_{ab} - D_a D_b \alpha - \alpha S_{ab} + \tfrac{1}{2}\alpha T h_{ab}$$

For completeness we rewrite the constraint equations from above:

$$R + K^2 - K_{ab}K^{ab} = 2\rho$$

$$D_b K - D_a K^a{}_b = t_b \quad .$$

We will also find extremely useful the traces of the evolution equations:

$$(\pounds_v - \pounds_\beta)\sqrt{h} = -\alpha K\sqrt{h} \qquad h = \det(h_{ab})$$
$$(\pounds_v - \pounds_\beta)K = \alpha K^2 + \alpha R - D^a D_a \alpha - \alpha(S - \tfrac{3}{2}T) \quad .$$

We will investigate these equations in more detail later. The constraint equations will define the initial value problem while the evolution equations will propagate the initial data into the future. The last two equations will be powerful tools for analyzing various coordinate conditions we may seek to improve on α and β_a.

Before leaving this chapter we turn briefly to the subject of invariants of the gravitational field. As mentioned above, the quantities K_{ab}, h_{ab}, K, $\mathcal{R}_{ab}$, etc. depend on the particular congruence selected through spacetime. Therefore, they cannot serve as invariant measures of the amplitude of the gravitational field. For this we need spacetime scalars built out of the Riemann tensor.

There are four functionally independent differential invariants of the Riemann or Weyl tensor in vacuum.[13] They may be expressed as two complex invariants I and J:

$$I = \psi_{ABCD}\psi^{ABCD} = \tfrac{1}{2}\{R_{abcd}R^{abcd} + iR^*_{abcd}R^{abcd}\}$$

$$J = \psi_{ABCD}\psi^{CDEF}\psi_{EF}{}^{AB} = \tfrac{1}{2}\{R_{abcd} + iR^*_{abcd}\}R^{cdef}R_{ef}{}^{ab} ,$$

where ψ_{ABCD} is the Weyl spinor and R^*_{abcd} is the dual of the Riemann tensor

$$R^*_{abcd} = \tfrac{1}{2}R_{ab}{}^{ef}\varepsilon_{efcd} .$$

Now we wish to split R_{abcd} into spatial quantities on S_t so we can evaluate I and J at the event defined by a point on S_t. This is a straightforward exercise in algebra. One takes the Weyl tensor $C_{abcd} = R_{abcd}$, written in terms of E_{ab}, B_{ab}, n_a, and ε_{abc}, and multiplies it out. The result is:

$$I = 4(E_{ab}E^{ab} - B_{ab}B^{ab}) + 8iE_{ab}B^{ab}$$

$$J = 8(E_{ab}E^{b}_{c}E^{ca} - 3B_{ab}E^{b}_{c}B^{ca}) - 8i(B_{ab}B^{b}_{c}B^{ca} - 3E_{ab}B^{b}_{c}E^{ca})$$

The simplicity of this result is as striking as is the similarity of I to the invariants of Maxwell theory; $E_aE^a - B_aB^a$ and E_aB^a. As Sommers has pointed out[14] this result can be represented in "spinor-like" form as:

$$I = 4W_{ab}W^{ab} \qquad J = 8W_{ab}W^{b}_{c}W^{ca}$$

where $W_{ab} = E_{ab} + iB_{ab}$.

These invariants can be written in terms of the 3-curvatures $\mathcal{R}_{ab}$ and K_{ab} by use of the field equations, which for a vacuum ($R_{ab} = 0$) imply:

$$E_{ab} = P_{ab} = -\mathcal{R}_{ab} - KK_{ab} + K_{ca}K^{c}_{b}$$

$$B_{ab} = M_{(ab)} = -D_mK_{n(a}\varepsilon^{mn}{}_{b)} = -D_mK_{na}\varepsilon^{mn}{}_{b} \qquad (M_{[ab]} = 0)$$

For example,[15]

$$\mathcal{R}e(I) = 4(E_{ab}E^{ab} - B_{ab}B^{ab})$$

$$= 4\{R_{ab}R^{ab} - 2K_{ca}R^{a}{}_{b}K^{bc} + 2K\mathcal{R}_{ab}K^{ab} + K^2K_{ab}K^{ab}$$

$$- 2KK_{ca}K^{a}{}_{b}K^{bc} + K_{ca}K^{ab}K_{bd}K^{dc}$$

$$- 2(D_{[m}K_{n]a})D^{m}K^{na}\}$$

Notice that α and β_a do not appear in these invariant expressions. This is again a manifestation of our freedom to choose α and β_a arbitrarily. One can rewrite $\mathcal{R}e(I)$ in several equivalent ways by using the Hamiltonian constraint ($\mathcal{R} + K^2 - K_{ab}K^{ab} = 0$) or by using the Cayley-Hamilton theorem[16] applied to K_{ab}. The latter procedure allows us to relate the 4th order invariant of K_{ab} to the 1st, 2nd, and 3rd order invariants:

$$K_{ab}K^{bc}K_{cd}K^{da} = \tfrac{1}{6}K^4 + \tfrac{4}{3}KK_{ab}K^{b}{}_{c}K^{ca} + \tfrac{1}{2}(K_{ab}K^{ab})^2 - K^2K_{ab}K^{ab} \quad .$$

Besides giving us a measure of the "strength" of the gravitational field, I and J also provide a local algebraic classification for the Weyl tensor.[13] If $I^3 \neq 6J^2$, the gravitational field is algebraically general. It is type {2,1,1} or {2,2} if $I^3 = 6J^2 \neq 0$ and type {3,1}, {4}, or {-} if $I = J = 0$.[14]

References

Chapter II

1. Our notation will basically follow Geroch (1972) except our α is his ϕ and we include a shift vector. This chapter owes a great deal to discussions with Paul Sommers and the use of unpublished calculations of his.
2. This covariant definition of β_a is due to Stachel (1962, 1969).
3. Richard Matzner (private communication).
4. See for example Stachel (1962, 1969) or Sachs (1964). James W. York, Jr. (1966, 1972) and Geroch (1972) also emphasize this approach. For a detailed mathematical treatment see Schouten (1954) or Yano (1957).
5. Wheeler (1964) p. 345.
6. Eisenhart (1949), Fourès-Bruhat (1956), Sachs (1964), Wheeler (1964), Israel (1966), MTW (1973).
7. Our notation for the parts of R_{abcd} is basically due to Estabrook and Wahlquist (1964) except our S_{ab} is their T_{ab} and our E_{ab} is their A_{ab}. Since we differ by factors of ½ and by signs from them, one cannot take over their equations verbatim. This form of R_{abcd} in terms of Q_{ab}, A_{abc}, P_{abcd} and C_{abcd} in terms of E_{ab} and B_{ab} is due to Sommers.
8. A similar splitting of C_{abcd} may be found in Hawking (1966) or Ellis (1971).

9. A detailed study of this abstract structure as well as a mathematical explanation of α and β_a may be found in Fischer and Marsden (1972).

10. See for example Pirani (1964), Weber (1961), or MTW (1973).

11. This is proportional to the DeWitt spray. See Fischer and Marsden (1972) for more on this point.

12. See Stachel (1962, 1969) and Fischer and Marsden (1972).

13. There are fourteen for the most general nonvacuum R_{abcd} (Geheniau and Debever 1956). For the case of a nonnull Maxwell field or of a perfect fluid there are nine (Witten 1959, Penrose 1960, Ehlers 1962). These may easily be constructed out of E_{ab}, B_{ab}, ρ, t_a, S_{ab} just as we have done for the vacuum.

14. P. Sommers (private communication).

15. James W. York, Jr. independently performed a similar calculation of $Re(I)$ in terms of (R_{ab}, K_{ab}) in 1971 (unpublished).

16. Greenberg (1970).

17. Note that for gravitational wave solutions (type {3,1} or {4}) I and J are zero, but the Weyl tensor is not zero. For type {-}, the spacetime is conformally flat and $C_{abcd} = R_{abcd} = 0$ in vacuum. We will mainly be concerned if I or $J \to \infty$ as this would indicate a physical singularity in the spacetime (as opposed to a poor

choice of coordinates). See Hawking and Ellis (1973) p. 260 for further discussion.

Appendix

Chapter II

Here we derive a number of useful formulae which we stated without proof in the text. The intention is not so much to include proofs for mathematical rigor, but rather to illustrate certain manipulations which recur frequently.

The primary objects given are a four dimensional metric g_{ab} of Lorentz signature (-+++) and a unit hypersurface orthogonal timelike vector field n_a ($n_a n^a = -1$). From g_{ab} one constructs a four dimensional covariant derivative ∇_a, a Riemann tensor R_{abcd}, a Ricci tensor $R_{ab} = R^c{}_{acb}$, a Riemann scalar $R = R^a{}_a$, and a Weyl tensor C_{abcd}. The Riemann tensor can be defined by the action of two covariant derivatives operating on a vector $R_{abcd}K^d = 2\nabla_{[a}\nabla_{b]}K_c$ where $A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba})$ and $A_{(ab)} = \frac{1}{2}(A_{ab} + A_{ba})$. One also has a Levi-Civita alternating tensor ε_{abcd}.

With n one can build a projection tensor $h_{ab} = g_{ab} + n_a n_b$ which acts as a metric tensor on the hypersurface S_t orthogonal to n. A trivial lemma gives:

<u>Lemma</u> $n^a h_{ab} = 0 \qquad h_{ab}h^{bc} = h_a{}^c \qquad h_{ab}h^{ab} = 3$

<u>Proof</u> $n^a h_{ab} = n^a(g_{ab}+n_a n_b) = n_b - n_b = 0$

$$h_{ab}h^{bc} = (g_{ab}+n_a n_b)(g^{bc}+n^b n^c)$$
$$= \delta_a{}^c + n_a n^c + n_a n^c - n_a n^c = h_a{}^c$$
$$h_{ab}h^{ab} = (g_{ab}+n_a n_b)(g^{ab}+n^a n^b) = 4 - 1 - 1 + 1 = 3$$

With h_{ab} one can define the extrinsic curvature tensor K_{ab} by:

$$K_{ab} = -h_a{}^c h_b{}^d \nabla_c n_d$$

That this tensor is spatial is obvious and that it is symmetric depends on $n_a = -\alpha \nabla_a t$.

Lemma $K_{ab} = K_{ba}$

Proof
$$K_{[ab]} = -h_{[a}{}^c h_{b]}{}^d \nabla_c n_d = -h_a{}^c h_b{}^d \nabla_{[c} n_{d]}$$
$$= h_a{}^c h_b{}^d \nabla_{[c} \alpha \nabla_{d]} t$$
$$= h_a{}^c h_b{}^d (\nabla_{[c}\alpha)(\nabla_{d]} t) \quad \text{since } \nabla_{[c}\nabla_{d]} t = 0$$
$$= -h_a{}^c h_b{}^d (\nabla_{[c} \ell n \alpha) n_{d]}$$
$$= 0 \quad \text{by } n \cdot h = 0$$

We can also produce a spatial covariant derivative D_a operating on spatial tensors $T^c{}_b$ by projecting the full covariant derivative:

$$D_a T^c{}_b = h_a{}^m h^c{}_n h_b{}^p \nabla_m T^n{}_p$$

As expected h_{ab} is covariantly constant with respect to D_a:

Lemma $D_a h_{cb} = 0$

Proof
$$D_a h_{ab} = h_a{}^m h_c{}^n h_b{}^p \nabla_m h_{np}$$
$$= h_a{}^m h_c{}^n h_b{}^p \nabla_m (g_{np} + n_n n_p)$$
$$= h_a{}^m h_a{}^n h_n{}^p \nabla_m n_n n_p = 0$$

The proper acceleration $a_b = n^c \nabla_c n_b$ of n can be expressed in terms of the lapse function α:

Lemma $a_b = D_b \ell n \alpha$

Proof $a_b = n^c \nabla_c n_b = 2n^c \nabla_{[c} n_{b]}$ since $n^c \nabla_b n_c = 0$

$= 2n^c (\nabla_{[c} \ell n \alpha) n_{b]}$ by previous lemma

$= 2n^c (D_{[c} \ell n \alpha) n_{b]} + 2n^c (n^d \nabla_d \ell n \alpha) n_{[a} n_{b]}$

$= n^c (D_c \ell n \alpha) n_b - n^c (D_b \ell n \alpha) n_c$

$= D_b \ell n \alpha$

This in turn allows us to express K_{ab} in terms of $\nabla_a n_b$ without the projection terms:

Lemma $K_{ab} = -\nabla_a n_b - n_a a_b = -\nabla_a n_b - n_a D_b \ell n \alpha$

Proof $K_{ab} = -h_a{}^c h_b{}^d \nabla_c n_d$

$= -(g_a{}^c + n_a n^c)(g_b{}^d + n_b n^d) \nabla_c n_d$

$= -\nabla_a n_b - n_a a_b - n_b n^d \nabla_a n_d - n_a n^c n_b n^d \nabla_c n_d$

$= -\nabla_a n_b - n_a a_b$

Corollary $K_{ab} = -h_a{}^c \nabla_c n_b \neq - h_b{}^d \nabla_a n_d$

We will use this result over and over in the remaining theorems.

Before deriving the evolution equations we prove two more lemmas about the acceleration.

Lemma $D_a a_b = \nabla_a a_b + n_b a^c K_{ca} + n_a n^c \nabla_c a_b$

Proof $D_a a_b = h_a{}^c h_b{}^d \nabla_c a_d$

$$= \nabla_a a_b + n_a n^c \nabla_c a_b + n_b n^d \nabla_a a_d + n_a n_b n^d n^c \nabla_c a_d$$

$$= \nabla_a a_b + n_a n^c \nabla_c a_b - n_b a^d \nabla_a n_d - n_a n_b a^d n^c \nabla_c n_d$$

$$= \nabla_a a_b + n_a n^c \nabla_c a_b - n_b a^d K_{ad} + n_b a^d n_a a_d - n_a n_b a^d a_d$$

$$= \nabla_a a_b + n_a n^c \nabla_c a_b + n_b a^d K_{ad}$$

Lemma $\alpha^{-1} D_a D_b \alpha = D_a a_b + a_a a_b$

Proof $D_a D_b + a_a a_b = D_a a_b \ell n\alpha + (D_a \ell n\alpha)(D_b \ell n\alpha)$

$$= D_a \alpha^{-1} D_b \alpha + \alpha^{-2} (D_a \alpha)(D_b \alpha)$$

$$= \alpha^{-1} D_a D_b \alpha$$

Now we introduce the Lie derivative along αn. Operating a tensor $T^a{}_b$ it is defined by:

$$\pounds_v T^a{}_b = v^c \nabla_c T^a{}_b - T^c{}_b \nabla_c v^a + T^a{}_c \nabla_b v^c$$

where v is some congruence (for instance αn). As a number of authors have pointed out, it is the proper geometric generalization of our notion of time derivative. Thus, it will be the operator we use on our field variables (h_{ab}, K_{ab}) in formulating the evolution equations.

Theorem $\pounds_{\alpha n} h_{ab} = -2\alpha K_{ab}$

Proof
$$\pounds_{\alpha n} h_{ab} = \alpha n^c \nabla_c h_{ab} + 2h_{c(a}\nabla_{b)}\alpha n^c$$
$$= \alpha[n^c \nabla_c n_a n_b - 2h^c{}_{(a}K_{b)c} - 2h^c{}_{(a}n_{b)}a_c]$$
$$= \alpha[2n_{(a}a_{b)} - 2K_{ab} - 2n_{(a}a_{b)}]$$

Theorem $\pounds_{\alpha n} K_{ab} = -\alpha K_{ca}K^c{}_b - D_a D_b \alpha - Q_{ab}$

Proof
$$\pounds_{\alpha n} K_{ab} = \alpha n^c \nabla_c K_{ab} + 2K_{c(a}\nabla_{b)}\alpha n^c$$
$$= \alpha[-n^c \nabla_c \nabla_a n_b - n^c \nabla_c n_a a_b - 2K_{c(a}K_{b)}{}^c - 2K^c{}_{(a}n_{b)}a_c]$$
$$= -\alpha[n^c \nabla_a \nabla_c n_b + n^c R_{cabd} n^d + n_a n^c \nabla_c a_b + a_b a_a + 2K_{c(a}K_{b)}{}^c + 2K^c{}_{(a}n_{b)}a_c]$$
$$= -\alpha[\nabla_a a_b - (\nabla_a n^c)(\nabla_c n_b) + Q_{ab} + n_a n^c \nabla_c a_b + a_b a_a + 2K_{c(a}K_{b)}{}^c + 2K^c{}_{(a}n_{b)}a_c]$$

$$= -\alpha[\nabla_a a_b + n_b a^c K_{ca} + n_a n^c \nabla_c a_b + a_a a_b + K_{ca} K_b{}^c + Q_{ab}]$$

$$= -\alpha[D_a a_b + a_a a_b + K_{ca} K_b{}^c + Q_{ab}]$$

$$= -D_a D_b \alpha - \alpha K_{ca} K_b{}^c - \alpha Q_{ab}$$

Our next task is to relate Q_{ab} by the field equations to either matter variables (ρ, t_a, S_{ab}) or three dimensional curvatures (K_{ab}, R_{ab}). Before that let us tabulate two last formulas using the Lie derivative:

$$\underline{\text{Lemma}}\quad \pounds_{\alpha n} n^a = -n^a n^c \nabla_c \alpha \qquad \pounds_{\alpha n} n_a = \alpha a_a - \nabla_a \alpha$$

Moving on to the splitting of the spacetime Riemann tensor, it is a simple matter to show that if:

$$R_{abcd} = 4n_{[a}Q_{b][c}n_{d]} + 2n_{[a}M_{b]}{}^e\varepsilon_{ecd} + 2n_{[c}M_{d]}{}^e\varepsilon_{eab} + P_{abcd}$$

then

$$Q_{bc} = n^a R_{abcd} n^d$$

$$M_d{}^e\varepsilon_{ebc} = A_{dbc} = h_b{}^a h_c{}^m h_d{}^n R_{amnp} n^p$$

$$P_{abcd} = h_a{}^m h_b{}^n h_c{}^p h_d{}^r R_{mnpr} \quad .$$

One only need be careful in the use of the antisymmetrization

convention and then use $n^a n_a = -1$ and $n^a Q_{ab} = n^a M_{ab} = n^a P_{abcd} = 0$.

On contracting R_{abcd} once we obtain the Ricci tensor:

$$R_{bd} = R^a{}_{bad} = Q_{bd} - P_{bd} - Q n_b n_d - 2M^{ae}\varepsilon_{ea(b}n_{d)}$$

where we have used

$$\varepsilon_{acd}\varepsilon^{bcd} = 2h_a{}^b \quad .$$

One further contraction gives the Riemann scalar:

$$R = -P + 2Q$$

Together they yield the Einstein tensor which can be equated to the matter tensor T_{ab}:

$$G_{ab} = R_{ab} - \tfrac{1}{2}g_{ab}R = T_{ab}$$

$$Q_{bd} - P_{bd} - Q n_b n_d - 2M^{ae}\varepsilon_{ea(b}n_{d)} - \tfrac{1}{2}(-P + 2Q)(h_{bd} - n_b n_d)$$

$$= \rho n_b n_d - 2n_{(b}t_{d)} + S_{bd}$$

Now projecting with $n^b n^d$, $n^b h^d{}_c$, and $h^b{}_c h^d{}_r$ we have:

$$\rho = -\tfrac{1}{2}P$$

$$t_a = h^{cb}M_c{}^e\varepsilon_{eba}$$

$$S_{cr} = Q_{cr} - P_{cr} - \tfrac{1}{2}(-P + 2Q)h_{cr}$$

Similarly, by placing the expression for R_{ab} and R_{abcd} in the formula relating C_{abcd}, R_{abcd}, and R_{ab} and then using the Weyl split in terms of E_{ab} and B_{ab}, we find after projecting that

$$E_{ab} = \tfrac{1}{2}(Q_{ab} + P_{ab}) + \tfrac{1}{6}(P + Q)h_{ab}$$

$$B_{ab} = M_{ab} - \tfrac{1}{2}\varepsilon_{acb}t^c$$

It is a simple task algebraically to invert these formulae to give the other expressions listed in the text.

Now we want to arrive at relations between the spacetime curvature and the space curvatures. These are fairly straightforward proofs using what we have done before plus the fact that $h_{[a}{}^m h_{b]}{}^n \nabla_m n_n = 0$.

<u>Theorem</u> $P_{abcd} = R_{abcd} + 2K_{c[a}K_{b]d}$

<u>Proof</u> $R_{abcd}K^d = 2D_{[a}D_{b]}k_c \qquad n\cdot k = 0$

$$= 2h_{[a}{}^m h_{b]}{}^n h_c{}^p \nabla_m h_n{}^r h_p{}^s \nabla_r k_s$$

$$= 2h_{[a}{}^{m}h_{b]}{}^{n}h_{c}{}^{p}\{h_{n}{}^{r}h_{p}{}^{s}\nabla_{m}\nabla_{r}k_{s} + (\nabla_{r}k_{s})\nabla_{m}h^{r}{}_{n}h^{s}{}_{p}\}$$

$$= h_{a}{}^{m}h_{b}{}^{r}h_{c}{}^{s}R_{mrst}k^{t} + 2h_{[a}{}^{m}h_{b]}{}^{n}h_{c}{}^{p}(\nabla_{r}k_{s})\{h_{n}{}^{r}\nabla_{m}h_{p}{}^{s} + h_{p}{}^{s}\nabla_{m}h_{n}{}^{r}n_{n}n^{r}\}$$

$$= h_{a}{}^{m}h_{b}{}^{r}h_{c}{}^{s}R_{mrst}k^{t} + 2h_{[a}{}^{m}h_{b]}{}^{r}h_{c}{}^{p}(n^{s}\nabla_{r}k_{s})\nabla_{m}n_{p}$$

$$= h_{a}{}^{m}h_{n}{}^{r}h_{c}{}^{s}R_{mrst}k^{t} - 2h_{[a}{}^{m}h_{b]}{}^{r}h_{c}{}^{p}(\nabla_{r}n_{s})(\nabla_{m}n_{p})k^{s}$$

$$= \{h_{a}{}^{m}h_{b}{}^{r}h_{c}{}^{s}R_{mrst} - 2h_{[a}{}^{m}h_{b]}{}^{r}h_{c}{}^{p}K_{rt}K_{mp}\}k^{t}$$

$$\mathcal{R}_{abcd} = h_{a}{}^{m}h_{n}{}^{r}h_{c}{}^{s}h_{d}{}^{t}R_{mrst} - 2K_{c[a}K_{b]d}$$

<u>Theorem</u> $A_{cab} = -2D_{[a}K_{b]c} = h_{a}{}^{m}h_{b}{}^{n}h_{c}{}^{p}n^{r}R_{mnpr}$

<u>Proof</u> $A_{cab} = h_{a}{}^{m}h_{b}{}^{n}h_{c}{}^{p}R_{mnpr}n^{r}$

$$= 2h_{a}{}^{m}h_{b}{}^{n}h_{c}{}^{p}\nabla_{[m}\nabla_{n]}n_{p}$$

$$= -2h_{a}{}^{m}h_{b}{}^{n}h_{c}{}^{p}\nabla_{[m}K_{n]p} - 2h_{a}{}^{m}h_{b}{}^{n}h_{c}{}^{p}\nabla_{[m}n_{n]}a_{p}$$

$$= -2D_{[a}K_{b]c}$$

<u>Theorem</u> $t_{b} = D_{b}K - D_{a}K^{a}{}_{b}$

<u>Proof</u> $t_{b} = h^{ca}A_{cab} = -2h^{cb}D_{[a}K_{b]c}$

$$= -D_{a}K_{b}{}^{a} + D_{a}K$$

Theorem $K_{ab}K^{bc}K_{cd}K^{da} =$

$$\tfrac{1}{6}K^4 + KK_{ab}K^b{}_cK^{ca} + \tfrac{1}{2}(K_{ab}K^{ab})^2 - K^2K_{ab}K^{cd}$$

Proof Our proof follows Greenberg (1970). Since K_{ab} is a spatial tensor we regard it as a 3 x 3 symmetric tensor with characteristic value equation:

$$|K_a{}^b - ph_a{}^b| = 0$$

$$p^3 - Ip^2 - Jp - K = 0$$

where $I = K^a{}_a$ $J = \tfrac{1}{2}[(K^{ab}K_{ab}) - K^2]$ $K = \det(K_{ab})$.

The Cayley-Hamilton theorem says that a tensor satisfies its own characteristic equation, so substituting K_{ab} for p yields:

$$K_{ab}K^b{}_cK^c{}_d - IK_{ab}K^b{}_d - JK_{ad} - Kh_{ad} = 0$$

The trace of this equation enables us to evaluate $\det(K_{ab})$ in terms of $K_{ab}K^b{}_cK^{ca}$, $K_{ab}K^{ab}$, and $K^a{}_a$:

$$\tfrac{1}{3}\{K_{ab}K^b{}_cK^{ca} - KK_{ab}K^{ab} - \tfrac{1}{2}[(K_{ab}K^{ab})^2 - K^2]K\} = \det(K_{ab})$$

Now multiply the characteristic equation by K^{da}

to get an equation relating the 4th order invariant $K_{ab}K^{bc}K_{cd}K^{da}$ to the 1st, 2nd, 3rd order invariants (having substituted for $\det(K_{ab})$). Notice that for $K = 0$ we simply have:

$$K_{ab}K^{bc}K_{cd}K^{da} = \tfrac{1}{2}(K_{ab}K^{ab})^2 \qquad (K = 0) \quad .$$

Chapter III

In the last chapter our emphasis was on the geometry of the spacelike hypersurfaces which we use to slice spacetime. We interpreted K_{ab} as the extrinsic curvature of S_t and a_a as the acceleration of the hypersurface orthogonal timelike congruence n_a. We will keep this reference system in what follows, but we will enlarge our investigation from the n_a congruence to the study of an arbitrary timelike congruence u_a which also intersects the slices.

The reason we wish to do this is not just a wish for theoretical completeness. As with every other part of this thesis, we are setting up the practical necessary machinery to enable us to construct spacetimes on a computer by evolving a given set of initial data off a spacelike hypersurface. As we have already seen, we may desire a nonnormal set of coordinates ($\beta_a \neq 0$) to label points of succeeding slices. Simultaneously with the n_a and v_a coordinate congruences we may have matter present (say a perfect fluid) with flow lines w_a.

This leads to a notion of constructing spacetime by specifying flows of several noninteracting interpenetrating fluids--some real fluids, some "coordinate" fluids.[1] This generic problem is usually only encountered in the specific. Should we take comoving coordinates? How does one interpret a tilted cosmology? Will the Schwarzschild throat "pinch-off"

if we use geodesic normal coordinates? What we will try to do here is set up a formalism in which these problems can be examined and compared in one framework. After studying the properties of a general timelike congruence we show how the ADM coordinate system naturally adapts itself to our coordinate congruence v_a.

For our general congruence u_a, we again introduce along with our timelike unit tangent u_a, a projection tensor h_{ab}, a Lie derivative $\pounds_u$, and a breakup of the covariant derivative of u_a:

$$u_a u^a = -1$$

$$h_{ab} = g_{ab} + u_a u_b$$

$$\nabla_a u_b = -u_a a_b + \theta_{ab} - \omega_{ab}$$

where $$a_a = u^b \nabla_b a_a$$

$$\theta_{ab} = h_a{}^c h_b{}^d \nabla_{(c} u_{d)}$$

$$\omega_{ab} = -h_a{}^c h_b{}^d \nabla_{[c} u_{d]}$$

As is well known,[2] θ_{ab} is the expansion tensor and ω_{ab} is the vorticity tensor of the congruence. One can further split θ_{ab} into a traceless shear tensor σ_{ab} and a trace θ

which is the volume expansion:

$$\theta_{ab} = \sigma_{ab} + \tfrac{1}{3}\theta h_{ab}$$

The slicing dependent tensors (u_a, a_a, σ_{ab}, θ, ω_{ab}) give a direct physical insight into the local nature of the congruence. As before, a_a measures the proper acceleration of u_a. If the volume increases or the lines expand away from each other, then $\theta > 0$ measures the isotropic part of that expansion. If the expansion possesses an anisotropic part, then σ_{ab} measures the distortion in different directions with the volume held fixed. Finally, the flow lines may be twisting or rotating so that they are no longer hypersurface orthogonal. In this case $\omega_{ab} \neq 0$ and the instantaneous axis of rotation as well as the magnitude of the vorticity is given by:

$$\omega^a = \tfrac{1}{2}\varepsilon^{abcd}u_b\omega_{cd}$$

$$\omega = (\tfrac{1}{2}\omega^{ab}\omega_{ab})^{\frac{1}{2}} = (\omega^a\omega_a)^{\frac{1}{2}} \quad .$$

We see immediately that the n_a congruence generated by our slicing surfaces S_t is nontwisting ($\omega_{ab} = 0$) and that the extrinsic curvature tensor K_{ab} is just the negative of the expansion tensor θ_{ab} for n_a. This establishes a very useful duality between the geometry of the spacelike slices S_t and their normal congruence n_a. For instance, the trace

K of the extrinsic curvature tensor (the mean curvature of $S_t^{(3)}$) is just $-\theta$, so that a positive mean curvature means that the normals to the surface are converging.[4]

In the case $\beta_a = 0$, the congruence of normals n_a is also the coordinate line congruence. However, when we choose a nonzero shift vector $\beta_a \neq 0$, a nonnormal coordinate line congruence v_a is generated in addition to the normal congruence n_a. Nonetheless, $K_{ab} = -\theta_{ab}$ where θ_{ab} is the expansion tensor of the n_a congruence. At the end of this chapter, we will study the relationship of the $\tilde{\theta}_{ab}$, $\tilde{\omega}_{ab}$, and $\tilde{a}_a$ tensors, formed from a unit vector $\hat{v}^a$ along the coordinate line congruence v^a, to the tensors of the n_a congruence. The point is that Einstein's equations are tied to the geometry of the spacelike slices S_t. This is one important reason we wish to study congruences (either coordinate line-v^a or matter flow-w^a) other than n_a.

We now look for the propagation equations satisfied by a general ω_{ab} and θ_{ab}.[5] As before the Lie derivative geometrizes the notion of time derivative and we find:[6]

$$\pounds_u u_a = a_a$$

$$\pounds_u h_{ab} = 2\theta_{ab}$$

$$\pounds_u \omega_{ab} = -D_{[a}a_{b]}$$

$$\pounds_u \theta_{ab} = (\theta^c{}_a + \omega^c{}_a)(\theta_{cb} + \omega_{cb}) + Q_{ab} + a_a a_b + D_{(a}a_{b)}$$

where as before $Q_{ab} = u^a R_{abcd} u^d$. When $\omega_{ab} = 0 = D_{[a}a_{b]}$ and we choose $u^a = \alpha n^a$, these equations reduce to the Einstein equations studied in the last chapter ($K_{ab} = -\theta_{ab}$ -- see Appendix).

Two aspects of this form for the propagation equations should be emphasized. First, the right hand sides are always spatial tensors, i.e. their contraction with u^a on any index vanishes. This is a consequence of the theorem that the Lie derivative of a spatial tensor is spatial (pointed out to the author by Paul Sommers). That is if

$$A_{ab}u^a = A_{ab}u^b = 0 \quad ,$$

then $u^a \pounds_u A_{ab} = u^b \pounds_u A_{ab} = 0$. The proof is given in the Appendix. Second, the Lie derivative term $\pounds_u$ subsumes terms which normally would appear on the right hand side.

This is to be contrasted to the method usually used in the study of congruences[7] where the time derivative is taken to be the covariant derivative along u_a:

$$\dot{A}_{ab} = u^c \nabla_c A_{ab} \quad .$$

Since this "time derivative" does not take spatial tensors into spatial tensors, one is forced to project the equations into the space orthogonal to u_a at each step. For example, the equations for ω_{ab} read:[7]

$$\dot{\omega}_{ab} = 2\omega_{c[a}\theta_{b]}{}^{c} + \nabla_{[a}a_{b]} + a^{c}\theta_{c[a}u_{b]} - a^{c}\omega_{c[a}u_{b]} - u_{[a}a_{b]}$$

$$\perp\dot{\omega}_{ab} \equiv h_{a}{}^{c}h_{b}{}^{d}\dot{\omega}_{cd} = 2\omega_{c[a}\theta_{b]}{}^{c} - h_{a}{}^{c}h_{b}{}^{d}\nabla_{[c}a_{d]} \quad .$$

Thus in the following we will always choose to work with $\pounds_{u}$ instead of $u^{c}\nabla_{c}$ (they agree if they are operating on scalars).

The next step is to obtain the equations for the propagation of the scalars of expansion, shear, and twist:

$$\theta = \theta^{a}{}_{a} \qquad \sigma^{2} = \tfrac{1}{2}\sigma_{ab}\sigma^{ab} \qquad \omega^{2} = \tfrac{1}{2}\omega_{ab}\omega^{ab} \quad .$$

These are written with the help of two other natural invariants (Greenberg-1970):

$$\chi = \det(\sigma_{ab}) = \sigma_{ab}\sigma^{b}{}_{c}\sigma^{ca}$$

$$\phi = \omega_{ab}\sigma^{b}{}_{c}\omega^{ca}$$

$$\pounds_{u}\theta = -\tfrac{1}{3}\theta^{2} - 2\sigma^{2} + 2\omega^{2} + \nabla^{c}a_{c} - \tfrac{1}{2}(\rho + S)$$

$$\pounds_{u}\sigma^{2} = -\tfrac{4}{3}\theta\sigma^{2} - 3\chi - \phi + \sigma^{ab}(E_{ab} + \tfrac{1}{2}S_{ab} + D_{a}a_{b} + a_{a}a_{b})$$

$$\pounds_{u}\omega^{2} = -\tfrac{4}{3}\theta\omega^{2} + 2\phi - \omega^{ad}D_{a}a_{b}$$

Greenberg also produces the propagation equations for χ, ϕ and interprets them geometrically. However, we shall not

pursue them further here.

The equation for $\pounds_u\theta$ is the famous Raychaudhuri equation which has figured so prominently in the singularity theorems of Hawking and Penrose.[8] It tells us how acceleration, expansion, shear, twist, and matter affect expansion. In particular, nonzero θ (expansion or convergence) and shear tend to damp expansion, but they drive convergence ($\theta < 0$). Vorticity tends to stop convergence and drive expansion as our intuition about centrifugal acceleration would predict. Acceleration can act either way depending on the sign of its divergence. If $u^a R_{ad} u^d = \frac{1}{2}(\rho + S) \geq 0$, then matter causes convergence. This condition is called the strong energy condition if it holds for every timelike u_a. For a perfect fluid:

$$T_{ab} = \rho u_a u_b + p h_{ab} = R_{ab} - \tfrac{1}{2} g_{ab} R$$

$$\rho + S = \rho + 3p \geq 0$$

and we assume this is true. The condition also holds for a Maxwell field.[8]

As one might expect, only the trace or isotropic part of the matter drives the isotropic convergence. When we turn to the propagation of the anisotropic or shear part of the expansion we find that the anisotropic stress $S_{ab} - \frac{1}{3} S h_{ab}$ drives it:

$$\pounds_u \sigma_{ab} = (\sigma^c{}_a + \omega^c{}_a)(\sigma_{cb} + \omega_{cb}) + E_{ab} + a_a a_b + D(a^a b)$$

$$+ \tfrac{1}{2}(S_{ab} - \tfrac{1}{3}Sh_{ab}) + \tfrac{1}{3}h_{ab}(2\sigma^2 - 2\omega^2 - \nabla^c a_c) \quad .$$

The electric components E_{ab} of the Weyl tensor also cause shear (which as a result drives convergence through σ^2). Thus, one says that a "pure" gravitational field causes astigmatic focusing of timelike lines.[9] Finally, we note that the vorticity tensor ω_{ab} feeds off of the antisymmetric spatial gradient of the acceleration.[10] However, from $\pounds_u \omega^2$ we see that if initially we have $\omega_{ab} = \omega^2 = 0$, then the vorticity vanishes for all time.[11]

Now all of this physical insight about how matter and curvature affect expansion and shear can be taken over directly to the study of Einstein's equations. Thus we say that the Weyl tensor causes the tracefree part of the extrinsic curvature to change or that matter density causes the mean curvature to grow. There is one very significant difference, however, between a congruence of matter lines and a coordinate line congruence.

In a spacetime with fluid flow the shear, expansion, twist, and acceleration of that fluid is determined by the physics. That is, there is one and only one congruence on the spacetime which describes the flow. In contrast, the coordinate freedom to choose α and β_a yields an infinite number of different coordinate line congruences through the same

spacetime. As O'Murchadha and York (1973) point out, this can be interpreted by saying the "slicer" of spacetime may choose at each event the acceleration vector $\tilde{a}_a$ of the coordinate congruence v^a. For example, locally we can "push" the coordinate lines apart to halt convergence, let $\alpha \to 0$ to slow down the evolution, or twist the lines to follow a rotating fluid. In this sense the shear, expansion, twist formalism as applied to a fluid is predictive while as applied to a coordinate congruence it is descriptive. Nonetheless, we will find this descriptive aspect very useful in sorting out various coordinate conditions.

We wish now to show how the ADM[12] coordinates fit naturally into the abstract index methods we have used up to this point. The time coordinate t will be chosen such that t = constant on each S_t. Then the metric tensor is split up into time and space components as follows:

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \qquad g^{ab} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix}$$

where i,j,k, etc. range from 1 to 3. These indices are raised with γ^{ij}:

$$g_{ab}g^{bc} = \delta_a{}^c \qquad \gamma_{ij}\gamma^{ik} = \delta_j{}^k$$

$$\beta^i = \gamma^{ij}\beta_j \quad .$$

At this point one can simply calculate[13] the four dimensional Christoffel symbols in terms of the three dimensional ones together with K_{ij}--the extrinsic curvature of the three space given by:

$$K_{ij} = -\frac{1}{2\alpha}(\gamma_{ij,o} - \beta_{i\cdot j} - \beta_{j\cdot i})$$

and then grind out the Riemann and Ricci tensors for the spacetime. With these in hand, one applies the Einstein field equations to fix the Ricci tensor. The result is a set of evolution and constraint equations in terms of γ_{ij} and K_{ij}.

Following the approach of this thesis, however, we already have the evolution and constraint equations written out (Chapter II) and we have but to take the i,j component of each of them to obtain the ADM equations. The only detail left is to verify that the aforestated correspondence:

$$D_a \leftrightarrow \cdot \qquad a = i$$

$$\pounds_v \leftrightarrow \partial_o$$

$$\left.\begin{aligned} K_{\alpha\beta} &\leftrightarrow K_{ij} \\ h_{\alpha\beta} &\leftrightarrow \gamma_{ij} \\ R_{\alpha\beta} &\leftrightarrow R_{ij} \end{aligned}\right\} \quad \alpha = i \quad \beta = j$$

$$\beta^a \leftrightarrow \beta^i \qquad a = i$$

holds where the left hand side lists quantities as defined in Chapter II and the right hand side lists quantities built up in three dimensions.

The proof is straightforward and is placed in the Appendix. The results are:

$$n_a = (-\alpha,0,0,0) \qquad n^a = \alpha^{-1}(1,-\beta^i)$$

$$a_a = \alpha^{-1}(\alpha_{,k}\beta^k,\alpha_{,i}) \qquad a^a = \alpha^{-1}(0,\alpha_{,}{}^i)$$

$$\beta_a = (\beta^k\beta_k,\beta_i) \qquad \beta^a = (0,\beta^i)$$

$$v_a = (-\alpha^2 + \beta^k\beta_k,\beta_i) \qquad v^a = (1,0,0,0)$$

$$\pounds_v = \partial_0 \qquad \pounds_\beta = \pounds_{\beta i}$$

where $\pounds_{\beta i}$ is a three dimensional Lie derivative along β^i. The equations in Chapter II become:

$$(\partial_0 - \pounds_{\beta i})\gamma_{ij} = -2\alpha K_{ij}$$

$$(\partial_0 - \pounds_{\beta i})K_{ij} = -2\alpha K_{\ell i}K_j{}^\ell + \alpha K K_{ij} + \alpha R_{ij} - \alpha_{.ij} + \alpha S_{ij} - \tfrac{1}{2}\alpha T\gamma_{ij}$$

$$R + K^2 - K_{ij}K^{ij} = 2\rho$$

$$K_{.i} - K^j{}_{i.j} = t_i$$

$$(\partial_0 - \pounds_{\beta i})K = \alpha K^2 + \alpha R - \alpha_{.i}{}^i - \alpha(S - \tfrac{3}{2}T)$$

$$(\partial_0 - \pounds_{\beta i})\sqrt{\gamma} = -\alpha K$$

with $K = K_{ij}\gamma^{ij}$, $T = -\rho + S^{i}{}_{i}$, and $\gamma = \det(\gamma_{ij})$.

These equations are essentially those of ADM (see p. 235-236 of Witten)[14] except we choose to retain K_{ij} as the basic field variable instead of using the momentum:

$$\pi^{ij} = -\sqrt{\gamma}(K^{ij} - g^{ij}K) \quad .$$

The above form is the one we will use for the computer since we can write the equations as 6 x 6 matrix equations for the components of γ_{ij} and K_{ij}.

As the last item of this chapter let us compare the shear, expansion, twist, and acceleration of the n_a and v_a congruences. We assume that a lapse function α has been chosen, but that the choice of the shift vector β_i is still at our disposal. If one chooses $\beta_i = 0$, then $v_a = \alpha n_a$ and we have a <u>normal</u> coordinate congruence. If $\beta_i \neq 0$, we say the coordinate system is <u>tilted</u>.[15] In the latter case we obtain a unit tangent vector for the $v^a = \frac{\partial}{\partial t}$ congruence:

$$\hat{v}^a = Q(1,\vec{0}) \qquad Q = (\alpha^2 - \beta_k\beta^k)^{-\frac{1}{2}} \quad .$$

Continuing to borrow the language of some cosmologists, we define the hyperbolic <u>angle of tilt</u> ψ to be:

$$\cosh\,\psi = -n_a\hat{v}^a = \alpha Q \quad .$$

We note that as $\beta^i \to 0$, $Q \to \alpha^{-1}$, and $\cosh \psi \to 1 \Rightarrow \psi \to 0$. The unit vector giving the direction of tilt (designated $\tilde{c}^a$ by King and Ellis 1973) is just the unit shift vector:

$$\tilde{c}^a = \hat{\beta}^a = (\beta_k \beta^k)^{-\frac{1}{2}} (0, \beta^i) \quad ,$$

that is the projection of $\hat{v}^a$ into the surface S_t divided by $\sinh \psi$:

$$(\sinh \psi)\tilde{c}^a = h^a{}_b \hat{v}^b \qquad h^a{}_b = \delta^a{}_b + n^a n_b$$

$$\sinh \psi = Q(\beta_k \beta^k)^{\frac{1}{2}} \quad .$$

The n^a congruence is twistfree with acceleration $a_b = D_b \ell n \alpha$, expansion $\theta = -K$, and shear $\sigma_{ab} = K_{ab} - \frac{1}{3} K h_{ab}$. This may or may not provide a coordinate system adapted to the physics of the problem at hand. The freedom to select any β_a we wish means we can give our coordinate congruence v^a acceleration, twist, expansion, and shear, different from the normal congruence. To illustrate this let us calculate the expansion $\tilde{\theta}$ of $\hat{v}^a$:

$$\tilde{\theta} = \nabla_a v^a = Q[\alpha\theta + \pounds_v \ell n(\cosh \psi) + D^a \beta_a] \quad .$$

From this we see that when $\theta = 0$ ($K = 0$ maximal slicing), the $\hat{v}^a$ coordinate congruence is generally expanding or converging.

This has practical consequences for computer applications since an easy quantity to follow along coordinate lines is the determinant of three metric $\gamma = \det(\gamma_{ij})$. Now from before[16] we know:

$$(\pounds_v - \pounds_\beta)\sqrt{\gamma} = -\alpha K\sqrt{\gamma} \qquad \text{(general)}$$

so if $K = 0$, then

$$\pounds_v \ell n\sqrt{\gamma} = \beta_{i.}{}^{i} \quad . \qquad (K = 0)$$

Thus, if we have a <u>tilted</u> coordinate system, the value of the determinant is <u>not</u> conserved along coordinate lines except in the special case $\beta_{i.}{}^{i} = 0$. This contrasts to the conservation law which obtains for a normal congruence ($\beta_a = 0$):

$$\pounds_{\alpha n} \ell n\sqrt{\gamma} = 0 \qquad (K = 0) \quad .$$

The importance of conservation laws such as the one above and such as the constraint equations is that they allow a check on the accuracy of computations as we shall see later. Of course, even in the tilted maximal system one can compare $\pounds_v \ell n\sqrt{\gamma}$ with $\beta_{i.}{}^{i}$ but this involves an extra differentiation.

This duality between $\tilde{\theta}$ and $\frac{\partial}{\partial t}\sqrt{\gamma}$ is as one would expect. If a congruence of coordinate lines is expanding one would think the determinant γ would be affected since the

proper volume element is $\gamma^{\frac{1}{2}}d^4x$. In fact, one can obtain a direct relation by combining the equations for $\tilde{\theta}$ and for $\pounds_v\sqrt{\gamma}$ to find:

$$\tilde{\theta} = Q\frac{\partial}{\partial t}\ell n(\alpha Q\gamma^{\frac{1}{2}})$$

Note that if $\beta_a = 0$, $\alpha Q = 1$ and

$$\tilde{\theta} \to \theta = -K = \alpha^{-1}\frac{\partial}{\partial t}\ell n(\gamma^{\frac{1}{2}}) \qquad (\beta_a = 0) \quad .$$

Otherwise, $\tilde{\theta} = 0$ only if

$$\frac{\partial}{\partial t}[\alpha\gamma^{\frac{1}{2}}(\alpha^2 - \beta_k\beta^k)^{-\frac{1}{2}}] = 0 \qquad (\tilde{\theta} = 0) \quad .$$

Similar analyses can be made for the acceleration, twist, and shear. The calculations are given in the Appendix:[17]

$$a_0 = 0 \qquad a_i = Q\partial_t(Q\beta_i) - Q^{-1}Q_{,i}$$

$$\tilde{\theta}_{ij} = \alpha Q\theta_{ij} + \beta_i\beta_j Q^2\dot{Q} + Q^3\beta_{(i}\dot{\beta}_{j)} + Q\beta_{(i\cdot j)} \qquad \dot{Q} = \partial_t Q$$

$$\tilde{\omega}_{ij} = 2Q_{,[i}\beta_{j]} + Q^3\beta_{[i}\dot{\beta}_{j]} - Q\beta_{[i\cdot j]}$$

See King and Ellis (1973) for a similar comparison between normal coordinate lines and tilted matter lines for spatially homogeneous cosmologies.

References

Chapter III

1. I thank William Press for stimulating discussions on this point.
2. See for instance Ellis (1971) and the references cited therein.
3. Eisenhart (1949) p. 168, Schouten (1954) p. 250.
4. For more on these sign conventions see MTW (1973) p. 511.
5. Our geometric approach follows Cattaneo's (1961). A very useful analysis of congruences which complements our approach is the dyadic analysis of Estabrook and Wahlquist (1964). For proofs and use of the Lie derivative see O'Murchadha (1973).
6. O'Murchadha (1973). His w_{ab} is the negative of ours.
7. See Ehlers, et al. (1961), Hawking (1966), Ellis (1971). For a complete set of these equations see Greenberg (1970).
8. See for example Hawking and Ellis (1973).
9. See Penrose (1968) for a similar analysis for null curves.
10. See Hawking (1966).
11. That is, w_{ab} is an invariant relation (IR) variable. Estabrook, Wahlquist, and Behr (1968).
12. For a summary see either ADM (1962) or MTW (1973).

13. Bryce DeWitt carried out this lengthy calculation at the start of our work. These notes were essential to the program we have pursued for the last two years. In this notation "." is the symbol for the three dimensional covariant derivative.

14. The equations given in Witten-ADM (1962) are for vacuum. They were extended to get general case ($T_{ab} \neq 0$) by B. Schutz (1972). They were written down in this form ($\beta^i = 0$) by Lichnerowicz (1944).

15. King and Ellis (1973).

16. James York first pointed out the importance of this equation to the author.

17. It has recently come to the author's attention that Wahlquist and Estabrook (1966) have made a similar analysis. Their ϕ and $\vec{A}$ roughly correspond to our Q and $\vec{\beta}$. However, their split of the metric tensor is just the inverse of the ADM split and so it is difficult to make a direct comparison.

Appendix

Chapter III

The derivation of the propagation equations for u_a, h_{ab}, and θ_{ab} proceeds exactly as in the Appendix for Chapter I except one uses:

$$\nabla_a u_b = -u_a a_b + \theta_{ab} - \omega_{ab}$$

instead of

$$\nabla_a n_b = -n_a a_b - K_{ab} \quad .$$

Also replace $\pounds_{\alpha n}$ by $\pounds_u$ and K_{ab} by $-\theta_{ab}$. Because of the anti-symmetry of ω_{ab} one can also use:

$$h_{c(a}\omega_{b)}{}^{c} = 0 \quad .$$

Although one can derive $\pounds_u \omega_{ab}$ from first principles as well, it may provide more insight into the difference between using $\pounds_u$ and $u^c\nabla_c$ for time derivatives if we take the equation for $\dot{\omega}_{ab} = u^c\nabla_c\omega_{ab}$ from, say, Greenberg (1970) page 99 and regroup the terms to yield our expression. Note that his $\theta = \frac{1}{3}\theta$ for us.

Theorem $\pounds_u \omega_{ab} = -D_{[a}a_{b]}$

Proof From Greenberg:

$$\dot{\omega}_{ab} = -\tfrac{2}{3}\theta\omega_{ab} + 2\sigma^{d}{}_{[a}\omega_{b]d} - \nabla_{[a}\dot{u}_{b]} + (\dot{u}^{d}\nabla_{d}u_{[a})u_{b]}$$

$$- u_{[a}\ddot{u}_{b]}$$

$$= 2\theta^{d}{}_{[a}\omega_{b]d} - \nabla_{[a}a_{b]} + (a^{d}\nabla_{d}u_{[a})u_{b]} - u_{[a}\dot{a}_{b]}$$

$$= 2\theta^{d}{}_{[a}\omega_{b]d} - \nabla_{[a}a_{b]} + a^{d}\theta_{d[a}u_{b]} - a^{d}\omega_{d[a}u_{b]}$$

$$- u_{[a}\dot{a}_{b]}$$

By adding and subtracting $2a^{d}\omega_{d[a}u_{b]}$ we can regroup the terms as follows:

$$\dot{\omega}_{ab} - 2\theta^{d}{}_{[a}\omega_{b]d} - 2a^{d}\omega_{d[a}u_{b]} =$$

$$-\nabla_{[a}a_{b]} + a^{d}\theta_{d[a}u_{b]} + a^{d}\omega_{d[a}u_{b]} - u_{[a}\dot{a}_{b]}$$

which yields

$$\pounds_u \omega_{ab} = -D_{[a}a_{b]} \quad ,$$

where we use:

$$\pounds_u \omega_{ab} = \dot{\omega}_{ab} - 2\omega_{d[a}\nabla_{b]}u^{d}$$

$$= \dot{\omega}_{ab} + 2\omega_{d[a}u_{b]}a^{d} - 2\omega_{d[a}\theta_{b]}{}^{d}$$

since $\omega_{d[a}\omega_{b]}{}^{d} = 0$

and where we expand our result from the last chapter:

$$D_a a_b = \nabla_a a_b + u_a \dot{a}_b - u_b a^d(\theta_{ad} - \omega_{ad})$$

$$= \nabla_a a_b + u_a \dot{a}_b - a^d(\theta_{da} + \omega_{da})u_b \quad .$$

We now show that the propagation equations for h_{ab}, ω_{ab}, and θ_{ab} reduce to the Einstein evolution equations for h_{ab} and K_{ab} when we take $\omega_{ab} = 0$, $\theta_{ab} = -K_{ab}$, and $\pounds_u = \pounds_{\alpha n}$. For $\pounds_u h_{ab}$ the result is immediate. Since $\pounds_u \omega_{ab} = 0$, we know $D_{[a}a_{b]} = 0$ and we can drop the symmetrization on $D_{(a}a_{b)}$ in $\pounds_u \theta_{ab}$. Finally changing signs for $\theta_{ab} = -K_{ab}$ we arrive at our previous evolution equations.

Next we prove the theorem that the Lie derivative of a spatial tensor is spatial.

<u>Theorem</u> If $A_{ab}u^a = A_{ab}u^b = 0$, then $u^a \pounds_u A_{ab} = u^b \pounds_u A_{ab} = 0$

<u>Proof</u> $u^a \pounds_u A_{ab} = u^a\{u^c \nabla_c A_{ab} + A_{cb}\nabla_a u^c + A_{ac}\nabla_b u^c\}$

$$= -A_{ab}u^c \nabla_c u^a + A_{cb}a^c = 0$$

where we used $u^a u_a = -1$ and $u^a(\theta_a{}^c - \omega_a{}^c) = 0$

The other contraction is proved similarly.

With the above results in hand we derive the propagation equations for θ and σ^2. We need the following lemma:

<u>Lemma</u> $\pounds_u h^{ab} = -2\theta^{ab} + 2u^{(a}a^{b)}$

<u>Proof</u> $\pounds_u h^{ab} = u^c\nabla_c h^{ab} - 2h^{c(a}\nabla_c u^{b)}$

$= u^c\nabla_c u^a u^b - 2h^{c(a}\theta_c^{b)}$

where we used $h^{ca}u_a = h^{c(a}\omega_c^{b)} = 0$.

<u>Theorem</u> $\pounds_u\theta = -\frac{1}{3}\theta^2 - 2\sigma^2 + 2\omega^2 + \nabla^c a_c - \frac{1}{2}(\rho + S)$

<u>Proof</u> $\pounds_u\theta = \pounds_u\theta_{ab}h^{ab} = h^{ab}\pounds_u\theta_{ab} + \theta_{ab}\pounds_u h^{ab}$

$= \theta_{ab}\theta^{ab} + \omega_{ab}\omega^{ab} + Q + a^c a_c + D^c a_c - 2\theta_{ab}\theta^{ab}$

$= -2\sigma^2 - \frac{1}{3}\theta^2 + 2\omega^2 - \frac{1}{2}(\rho + S) + \nabla^c a_c$

To derive the equation for $\pounds_u\sigma_{ab}$ we use the additive property of Lie derivatives:

$$\pounds_u\sigma_{ab} = \pounds_u\theta_{ab} - \tfrac{1}{3}\pounds_u\theta h_{ab}$$

$$= \pounds_u\theta_{ab} - \tfrac{1}{3}h_{ab}\pounds_u\theta - \tfrac{1}{3}\theta\pounds_u h_{ab}$$

$$= \pounds_u\theta_{ab} - \tfrac{1}{3}h_{ab}\pounds_u\theta - \tfrac{2}{3}\theta\theta_{ab}$$

and simply add the preceding propagation equations for θ_{ab} and θ. Replace all θ_{ab} by $\sigma_{ab} + \frac{1}{3}\theta h_{ab}$ and collect terms to finish the proof.

To obtain the equation for $\pounds_u\sigma^2$ we proceed as follows:

$$\pounds_u\sigma^2 = \tfrac{1}{2}\pounds_u\sigma_{ab}h^{ac}h^{bd}\sigma_{bd}$$

$$= \sigma^{ab}\pounds_u\sigma_{ab} + \sigma^{c}{}_{b}\sigma_{cd}\pounds_u h^{bd}$$

$$= \sigma^{ab}\pounds_u\sigma_{ab} - 2\sigma^{c}{}_{b}\sigma_{cd}\theta^{bd}$$

$$= \sigma^{ab}\pounds_u\sigma_{ab} - 6\chi - \tfrac{4}{3}\theta\sigma^2$$

One then plugs in $\pounds_u\sigma_{ab}$ and reduces terms using:

$$\sigma^{ca}\omega_{ab}\sigma^{b}{}_{c} = 0 \qquad \text{since} \quad \sigma^{ca}\sigma^{b}{}_{c} = \sigma^{c(a}\sigma^{b)}{}_{c}$$

$$\sigma^{ca}h_{ca} = 0$$

$$\sigma^{cb}D_{(c}a_{b)} = \sigma^{cb}D_{c}a_{b}$$

The result follows immediately. A similar derivation will give $\pounds_u\omega^2$.

Now we introduce the ADM coordinates and show that the i,j = 1,2,3 components of K_{ab} is indeed K_{ij} as defined by:

$$K_{ij} = -\frac{1}{2\alpha}(\gamma_{ij,0} - \beta_{i\cdot j} - \beta_{j\cdot i}) \quad .$$

To do this we use the relation (York 1973):

$$\bar{\Gamma}^{a}_{bc} = h^{r}_{b} h^{s}_{c} \Gamma^{a}_{rs} + n^{a} K_{bc}$$

between the Christoffel symbols formed from $h_{ab}(\bar{\Gamma})$ and those formed from $g_{ab}(\Gamma)$, to prove the following Lemma.

<u>Lemma</u> If $n^{a}k_{a} = 0$ $\quad D_{a}k_{b} = k_{j\cdot i}$ for a = i and b = j

<u>Proof</u> Since n_a is a unit vector normal to the t = constant surface we can represent it by (Wheeler 1964):

$$n_{a} = (-\alpha, \vec{0}) \qquad n^{a} = (\alpha^{-1}, -\alpha^{-1}\beta^{i})$$

and thus if $k^{d}n_{d} = 0$ we must have

$$k^{d} = (0, k^{i})$$

$$k_{d} = (k^{m}\beta_{m}, k_{i}) \quad .$$

As we proved for a_b we can prove for k_d that:

$$D_{a}k_{d} = \nabla_{a}k_{d} + n_{a}n^{c}\nabla_{c}k_{d} + n_{b}k^{d}K_{ad} \quad .$$

If we take the i,j = 1,2,3 component, $n_i = n_j = 0$ and

$$D_i k_j = \nabla_i k_j$$

$$= k_{j,i} - \Gamma_{ij}{}^a k_a = k_{j,i} - \Gamma_{ij}{}^a k_a - \Gamma_{ij}{}^m k_m$$

Now we need to replace Γ by $\bar{\Gamma}$. To do so we multiply by n_a:

$$n_a \bar{\Gamma}_{bc}{}^a = 0 = h_b{}^r h_c{}^s \Gamma_{rs}{}^a n_a - K_{bc}$$

since $\bar{\Gamma}$ is spatial.

Taking the i,j component and using $h_i{}^j = \gamma_i{}^j$, $h_i{}^a = 0$:

$$-\alpha \Gamma_{ij}{}^0 = K_{ij} \quad .$$

To get $\Gamma_{jk}{}^i$ we multiply by $h_a{}^i$ and use:

$$h_0{}^i = n_0 n^i = \beta^i$$

$$h^i{}_a \bar{\Gamma}_{jk}{}^a = h_j{}^r h_k{}^s \Gamma_{rs}{}^a h_a{}^i$$

$$\bar{\Gamma}_{jk}{}^i = \Gamma_{jk}{}^i - \alpha^{-1} \beta^i K_{jk}$$

Now we go back to our expression for $D_i k_j$:

$$D_i k_j = k_{j,i} + \alpha^{-1} K_{ij} k^m \beta_m - \Gamma_{ij}{}^m k_m$$

$$-\alpha^{-1} K_{ij} \beta^m k_m$$

$$= k_{j \cdot i}$$

Now we need two lemmas on Lie derivatives:

Lemma If $v^a = (1,0,0,0)$, then $\pounds_v = \partial_0$.

Proof Consider $\pounds_v A_{ab} = v^c \nabla_c A_{ab} + A_{cb}\nabla_a v^c + A_{ac}\nabla_b v^c$

$$= \nabla_0 A_{cb}$$

But we can always replace covariant derivatives with ordinary derivatives in a Lie derivative.

Lemma $\pounds_\beta h_{ab} = 2\nabla_{(a}\beta_{b)} + 2\beta^c a_c n_a n_b$

Proof $\pounds_\beta h_{ab} = \beta^c \nabla_c h_{ab} + 2h_{c(a}\nabla_{b)}\beta^c$

$$= 2n_{(a}\beta^c\nabla_c n_{b)} + 2\nabla_{(a}\beta_{b)} - 2\beta^c n_{(a}\nabla_{b)} n_c$$

$$= -2n_{(a}\beta^c K_{cb)} + 2\nabla_{(a}\beta_{b)} + 2\beta^c n_a n_b a_c$$

$$+ 2\beta^c n_{(a}K_{b)c}$$

Lemma $D_a\beta_b = \nabla_a\beta_b + n_a n^c \nabla_c \beta_b + n_b \beta^d K_{ad}$

Proof The same as for $D_a a_b$ in Appendix (Chapter II).

Putting all the above machinery together we prove:

Theorem $K_{ab} = -\frac{1}{2\alpha}(\gamma_{ij,0} - \beta_{i\cdot j} - \beta_{j\cdot i})$ for $a = i$, $b = j$.

Proof $(\pounds_v - \pounds_\beta)h_{ab} = -2\alpha K_{ab}$

$$\partial_0 h_{ab} - 2D_{(a}\beta_{b)} + 2n_a n^c \nabla_c \beta_b + 2n_b \beta^d K_{ad} = -2\alpha K_{ab}$$

Take the $a = i$, $b = j$ components and use $n_i = n_j = 0$, $h_{ij} = \gamma_{ij}$.

Since β^a is a spatial vector we have $\beta^a = (0,\beta^i)$. This yields $v^a = \alpha n^a + \beta^a = (1,-\beta^i) + (0,\beta^i) = (1,\vec{0})$, so we were justified in using $\pounds_v \to \partial_0$.

Now we can justify the statement used in the last chapter that the invariants formed from K_{ab} (such as K, $K_{ab}K^{ab}$, etc.) are the same as those formed from K_{ij}. We simply note that we use h^{ab} to raise indices in all these traces, but we find:

$$h^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix}$$

so that every trace of K_{ab} picks out only the K_{ij} part.

We remark finally that the Ricci tensor for S_t reduces as it should to the R_{ij} formed out of γ_{ij}. An easy way to see this is to define $R_{ijmn}k^n = 2k_{m\cdot[ji]}$ where $i,j,m,n =$

1,2,3. Then since $R_{abcd}K^d = 2D_{[a}D_{b]}K_c$ for $K^a n_a = 0$, we can use our previous results that $K^d = (0,K^i)$ and $D_i \to "\cdot"$ to take the a = i, b = j, c = m, d = n components and note the agreement. Then taking a trace and using our result on h^{ab} yields $R_{ab} \to R_{ij}$ for a = i, b = j.

Finally, we list the relations between the expansion, shear, twist, and acceleration of the $\hat{v}^a$ congruence and the n^a congruence. Normalizing $v^a = (1,\vec{0})$ yields:

$$\hat{v}^a = Q(1,\vec{0}) \qquad Q = (\alpha^2 - \beta_k\beta^k)^{-\frac{1}{2}}$$

$$\hat{v}_a = (-Q^{-1},Q\beta_i) \quad .$$

We know $\tilde{a}_0 = 0$ since $\hat{v}^b\tilde{a}_b = 0$ and we can calculate

$$\tilde{a}_i = Q\partial_t(Q\beta_i) - Q^{-1}Q_{,i} \quad .$$

Again, we know $v_{00} = v_{0i} = v_{i0} = 0$ where $v_{ab} = \theta_{ab} - \omega_{ab}$. To get the nonvanishing part we use:

$$v_{ij} = \nabla_i\hat{v}_j + \hat{v}_i\tilde{a}_j \quad .$$

But $\nabla_i\hat{v}_j = \hat{v}_{j,i} - \Gamma_{ij}{}^a\hat{v}_a$ which we can use to find

$$\nabla_i\hat{v}_j = (Q\beta_j)_{\cdot i} - \alpha QK_{ij} \quad .$$

Adding $\nabla_i \hat{v}_j$ to $\hat{v}_i \tilde{a}_j$ gives:

$$v_{ij} = 2Q_{[i}\beta_{j]} - \alpha Q K_{ij} + Q^2 \beta_i \partial_t (Q\beta_j) + Q\beta_{j\cdot i}$$

and taking symmetric and antisymmetric parts:

$$\tilde{\theta}_{ij} = \alpha Q \theta_{ij} + \beta_i \beta_j Q^2 \partial_t Q + Q^3 \beta_{(i}\dot{\beta}_{j)} + Q\beta_{(i\cdot j)} \quad \text{where } \dot{\beta}_j = \partial_t \beta_j$$

$$\tilde{\omega}_{ij} = 2Q_{[i}\beta_{j]} + Q^3 \beta_{[i}\dot{\beta}_{j]} - Q\beta_{[i\cdot j]}$$

$$\tilde{\theta} = \alpha Q \theta + Q\beta_{i\cdot}{}^{i} + \dot{Q} + Q\partial_t \ell n \alpha \quad \text{using } \dot{Q} = \partial_t (\alpha^2 - \beta_k \beta^k)^{-\frac{1}{2}}$$

We remark that we could have written $\tilde{\omega}_{ij}$ down straightaway using Cattaneo's (1959) formula:

$$\tilde{\omega}_{ij} = \hat{v}_0 \left[\tilde{\partial}_i \left(\frac{\hat{v}_j}{\hat{v}_0} \right) - \tilde{\partial}_j \left(\frac{\hat{v}_i}{\hat{v}_0} \right) \right] \quad \text{where } \tilde{\partial}_i = \partial_i - \left(\frac{\hat{v}_i}{\hat{v}_0} \right) \partial_0 \, .$$

Chapter IV

Now that we have set up the mathematical framework of Einstein's equations we will take a closer look at the problems caused and insights gained by trying to apply this framework to particular examples. First, we show how to set up the initial data and separate physics from coordinates. Then we sketch the notions of the Cauchy development from a given initial configuration. In passing we describe some aspects we hope to monitor of the physics of strong gravitational fields, such as horizons, singularities, and radiation. We find that the choice of time coordinate (i.e. α) is critical if we wish to observe this physics, and so we must consider the various slicing conditions we might employ. Finally, we examine slicing of several well known spacetimes (Schwarzschild, Kerr, homogeneous cosmologies, and collapsing spherical stars) which exhibit various of these structures. This will take us through Chapters IV, V, and VI.

There is probably no paper in the literature that is closer in spirit to our point of view than Lichnerowicz's great 1944 essay on "L'intégration des équations de la gravitation relativiste et le probleme des n corps." He sets up most of the equations we will face (the conformal scale equation, the maximal slicing equation, the evolution equations), analyzes the maximal initial value and evolution problems, studies existence and uniqueness of Cauchy

developments, and then sets up the two body problem with matter in a way completely analogous to the two vacuum black hole problem we will study later. I hope to indicate some of the progress which has been made in these areas in the last thirty years.

The conformal approach[1] to the initial value problem has given us a fairly clear physical picture of what it means to "solve" the constraint equations we derived in Chapter II. Also there is now a practical "recipe" for generating initial data corresponding to a given physical situation. For a review of many of these points see MTW Chapter 21.

The basic problem has always been to find "where the physics lies" in the maze of coordinate transformations and constraint equations. Recall that our dynamical variables are the spatial tensors K_{ab} and h_{ab} each with six functions of three variables. As Sachs (1964) points out, since we can impose four coordinate conditions (say on α and β_a) and since the set (h_{ab}, K_{ab}) are interrelated by four constraint equations, there are only 12-4-4 = 4 real dynamical degrees of freedom in a gravitational field. As gravity in some sense is a massless spin-2 field, we expect this result. Certainly for gravitational radiation one has only two helicity or polarization states for each part of the field h_{ab} and K_{ab}. However, when we set an initial data set, what are the four independent quantities we specify?

Lichnerowicz had much of the answer in his paper, but the full physical picture was not worked out until recently.[1] First we consider the metric tensor h_{ab}. We know that three of its functions describe the type of three-dimensional coordinate system we choose to label the point in each hypersurface. We say that we want to specify a <u>three geometry</u> represented by some metric tensor. Thus the metric tensors: diag $(1,1,1)$, diag $(1,1,\rho^2)$, and diag $(1,r^2,r^2 \sin^2 \theta)$ all represent the same flat three geometry but in cartesian, cylindrical, and spherical coordinate systems.

However, this leaves three functions to specify--one too many.

Therefore, we freely specify the metric only up to a conformal factor ϕ and then let the Hamiltonian constraint determine this factor. As York has repeatedly stressed, this means that we only specify the <u>conformal three geometry</u>. This can be deceptively simple. By just looking at the metric tensor components for the conformal 3-geometry we conclude that a closed Friedman universe, one or many black holes, and flat space all belong to the same conformal geometry class, i.e. conformally flat. This is not true, however, since one must also specify the global topology of the manifold on which the Hamiltonian constraint is to be solved, i.e. a 3-sphere, two asymptotically flat spaces connected by Einstein-Rosen bridges, or Euclidean three-space in the above cases. Having set the topology and the conformal geometry we turn to

the momenta.

Here we have the three momentum constraints to reduce by half the original six functions in K_{ab}. The trace K is the object we can then choose as a gauge condition.[2] A choice of particular value is to take $D_a K = 0$ so that the momentum constraints reduce to $D_a K^a{}_b = 0$. Then one solves these three equations to make K_{ab} covariantly transverse with respect to h_{ab}. At this point one has in hand two "bare" tensors (h_{ab}, K_{ab}). Using them in the Hamiltonian constraint we solve for ϕ which "dresses" (h_{ab}, K_{ab}) by the rule:

$$h'_{ab} = \phi^4 h_{ab} \ , \quad K'_{ab} = \phi^{-10} K_{ab} \ .$$

What makes the method work is that (h'_{ab}, K'_{ab}) now satisfy the constraints themselves, even though we only plugged in (h_{ab}, K_{ab}) to find them.

Recounting, we specified a conformal geometry (and topology) in a particular coordinate system and a transverse symmetric momentum with specified trace; then we solved for a conformal factor to take these two bare tensors (h_{ab}, K_{ab}) to our final "dressed" initial data (h'_{ab}, K'_{ab}). We will now briefly detail the mathematics of this procedure and then discuss the physics it describes.

Having freely chosen h_{ab}, we pick some symmetric tensor K_{ab}. From the work of Deser and York[3] we can now covariantly decompose K_{ab} into its trace (T), longitudinal (L),

and transverse traceless (TT) parts:

$$K^{ab} = K^{ab}{}_{TT} + K^{ab}{}_{L} + K^{ab}{}_{T}$$

$$K^{ab}{}_{L} = 2[D^{(a}W^{b)} - \tfrac{1}{3}h^{ab}D_{c}W^{c}]$$

$$K^{ab}{}_{T} = \tfrac{1}{3}h_{cd}K^{cd}h^{ab}$$

$$K^{ab}{}_{TT} = K^{ab} - K^{ab}{}_{T} - K^{ab}{}_{L}$$

Plugging this into the momentum constraint equation (done here for vacuum) yields:

$$D_{b}(LW)^{ab} = D_{b}(K^{ab} - K^{ab}{}_{T}) \quad ,$$

three second order, linear, self adjoint, strongly elliptic partial differential equations for W^{a}. The source term on the right hand side is the tracefree part of the "bare" tensor K^{ab}. The left hand side can be expanded[4] to read:

$$D_{b}(LW)^{ab} = D_{b}D^{b}W^{a} + \tfrac{1}{3}D^{a}D_{b}W^{b} + R^{a}{}_{b}W^{b} \quad ,$$

where again D_{a} and $R^{a}{}_{b}$ are formed from the bare h_{ab}. These equations can be solved numerically for W^{a} given appropriate boundary conditions.

With W^{a} in hand, we can now form $K^{ab}{}_{TT}$ by subtracting off $(LW)^{ab}$ and $K^{ab}{}_{T}$ from K^{ab}. We assume from now on that we wish to solve the initial value problem for asymptotically

flat space and that we will choose the maximal gauge:

$$K^{ab}{}_{T} = h^{ab}K_{ab} = K = 0 \qquad \text{(maximal)} \quad .$$

This is again in keeping with the original (Lichnerowicz) and latest (York) procedure. We will physically motivate this choice shortly. For cosmologies one would choose $D_a K = 0$ although $D_a K \neq 0$ can be handled.[5]

Now we use $(h_{ab}, K^{ab}{}_{TT})$ to set up the Hamiltonian constraint as a quasilinear elliptic differential equation for ϕ. To do this one needs to find how each term in:

$$R - K_{ab}K^{ab} = 0 \qquad (K = T^{ab} = 0)$$

transforms under a conformal transformation:

$$h'_{ab} = \phi^4 h_{ab} \quad .$$

From classical differential geometry one finds:[6]

$$\Gamma'^{a}{}_{bc} = \Gamma^{a}{}_{bc} + 2(\delta^{a}{}_{b}D_{c}\ell n\phi + \delta^{a}{}_{c}D_{b}\ell n\phi - h_{bc}D^{a}\ell n\phi)$$

$$R'^{ab}{}_{cd} = \phi^{-4}R^{ab}{}_{cd} + \delta^{[a}{}_{[c}\Omega^{b]}{}_{d]}$$

$$\Omega^{a}{}_{b} = 8\phi^{-6}[3\phi_{,}{}^{a}\phi_{,b} - \phi\phi_{;b}{}^{a} - \phi_{,c}\phi_{,}{}^{c}\delta^{a}{}_{b}]$$

$$R'^{a}{}_{b} = \phi^{-4}R^{a}{}_{b} + \tfrac{1}{4}[(n-2)\Omega^{a}{}_{b} + \delta^{a}{}_{b}\Omega_{c}{}^{c}]$$

$$R' = \phi^{-4}R + \tfrac{1}{2}(n-1)\Omega_{c}{}^{c}$$

We choose to transform K_{ab} by

$$K'_{ab} = \phi^{-10} K_{ab} \quad .$$

Thus, conformally mapping the Hamiltonian constraint

$$R' - K'_{ab} K'^{ab} = 0$$

yields the scale equation:[7]

$$\Delta\phi = \tfrac{1}{8} R\phi - \tfrac{1}{8} (K_{ab}{}^{TT} K^{ab}{}_{TT}) \phi^{-7}$$

which is a quasilinear elliptic equation for ϕ. Solutions exist and are unique in almost all cases of interest.[8]

We are now finished because after conformally "dressing" our (h_{ab}, K_{ab}) to (h'_{ab}, K'_{ab}) we find that the momentum constraint is still satisfied and that K'_{ab} is still TT. That is, the conformal technique not only converts the four constraint equations into elliptic equations with very "nice" properties, but it also decouples the problems of solving the momentum and Hamiltonian constraints. Furthermore, this technique has been extended to the most general case of $D_a K \neq 0$, $T^{ab} \neq 0$ and thus can handle any initial value problem we pose on compact or asymptotically flat manifolds. Care must be taken to choose the proper conformal mapping for K and T^{ab} in such cases. For details see

O'Murchadha (1973) and O'Murchadha and York (1973).

Physically, the procedure is also very appealing. First we set our gauge by choosing a coordinate system for the conformal geometry and a time coordinate through K. Then we specify a momentum distribution (K_{ab}) and solve for W^a. This vector turns out to be just the object that we need to measure the linear and angular momentum by surface integrals at spatial infinity.[9] Finally, we solve for ϕ and using it we read off the mass at spatial infinity.[10] Thus, each object in the theory has both a mathematical use and a physical interpretation.

Further, from a deeper study of the Hamiltonian structure Dirac[11] found that the trace K and $\ell n \gamma^{1/2}$ are canonically conjugate variables. One can regard ϕ as giving $\gamma^{1/2}$ since we can freely specify $\gamma^{1/2}$ of the bare metric h_{ab} and then find the dressed $\gamma' = \phi^6 \gamma^{1/2}$. The interpretation of these two variables in terms of "time and the Hamiltonian" has been much discussed. Misner[12] favored using $\gamma^{1/2}$ as intrinsic time in his studies of mixmaster universes while Kuchař[13] and York[14] have favored using $K \neq 0$ for extrinsic time. Therefore, from the Hamiltonian point of view, the choice of K and ϕ as the "extra $\frac{1}{3}$" of the variables in (h_{ab}, K_{ab}) is quite natural.

Since the conformal geometry seems to be central to the initial value problem, one would like to have a geometric object, like the Weyl tensor for spacetime, which

would be conformally invariant and useful for classifying the conformal classes of initial data. The problem is that the Weyl tensor identically vanishes for 3-spaces, leaving R_{abcd} entirely determined by R_{ab}. Fortunately, classical differential geometers[6] filled in the n = 3 gap by looking at the Bianchi identities, which read:

$$C^{a}{}_{bcd;a} = \frac{2(n-3)}{(n-2)} \left(R_{b[d;c]} - \frac{1}{2(n-1)} g_{b[d} R_{;c]} \right) ,$$

and noting that while the left hand side vanishes for n = 3 because $C_{abcd} = 0$, the right hand side vanishes from the combinatorical factor leaving:

$$R_{bdc} = R_{b[d;c]} - \tfrac{1}{4} g_{b[d} R_{;c]}$$

nonzero in general. The symmetries of R_{bdc} are as follows:

$$R^{a}{}_{ac} = R_{a(bc)} = R_{[abc]} = 0$$

which means R_{abc} has five independent components. Its importance lies in the fact that $R_{abc} = 0$ is equivalent to the condition that the 3-space be conformally flat.

York[15] has realized the importance of this tensor for general relativity and has pointed out a number of interesting properties of R_{abc}. First he converts it to a symmetric, covariantly transverse, traceless tensor β_{ab}:

$$\beta_{ab} = -\tfrac{1}{2}\varepsilon^{aef}R^{b}{}_{ef}$$

where we follow York's notation of ε^{aef} as a tensor density of weight +1. Then he forms the density of weight $\frac{5}{3}$ which is conformally invariant:

$$\tilde{\beta}^{ab} = \gamma^{\frac{1}{3}}\beta^{ab} \quad .$$

Together with the $\frac{5}{3}$ weight density $\tilde{\pi}^{ab}$:

$$\tilde{\pi}^{ab} = \gamma^{\frac{1}{3}}\pi^{ab} = -\gamma^{\frac{5}{6}}(K_{ab} - Kh_{ab}) \quad ,$$

the pair $(\tilde{\beta}^{ab}, \tilde{\pi}^{ab})$ form a spin-2 representation of the conformal geometry. That is, they are freely specifiable, symmetric, covariantly transverse, traceless, and conformally invariant. There is much work yet to be done on this remarkable tensor β_{ab} since it should have a natural relation to gravitational radiation. It needs to be calculated also for various exact solutions in order to gain further intuition as to its properties.

We will show in a later paper[17] how one can use the above conformal technique to obtain a "generalized superposition principle" for the initial value problem. Say one wishes to have an initial slice with two black holes moving towards each other. This is a non-time symmetric ($K_{ab} \neq 0$) and non-conformally flat ($\beta_{ab} \neq 0$) situation. The procedure

is to take two slices of Schwarzschild which are "boosted" (one with +v, the other with -v), then add their metrics and extrinsic curvatures. These will form the bare (h_{ab}, K_{ab}). Now make K_{ab} traceless and solve for the W^a which gives us $K_{ab}{}^{TT}$. Plug this in the scale equation, solve for ϕ, and then map $(h_{ab}, K_{ab}{}^{TT})$ onto $(h'_{ab}, K'_{ab}{}^{TT})$. The idea is to use physics to provide a "good guess" at a desired initial configuration, then use the conformal technique to "fine tune" the geometry to satisfy the constraint equations.

In closing this review of the initial value problem, we will point out a number of special cases which have been studied. All of them occur on maximal slices. The first, by Lichnerowicz, is the time symmetric situation ($\pi^{ab} = 0$). He set up the n body problem on a slice which was also conformally flat ($\beta_{ab} = 0$), where he used regions of matter as models for the bodies. The problem is particularly nice here since the momentum constraints are trivially satisfied and the Laplacian in the scale equation is a flat space Laplacian. Since then a number of such solutions have been analyzed: gravitational waves,[18] one or many black holes,[19] and cosmologies.[20] Many of these have $\beta_{ab} \neq 0$. O'Murchadha and York[21] have recently proved that of all maximal initial data sets in a given conformal class (some $\tilde{\beta}_{ab}$), the time symmetric slice has the least energy.

A generalization of time symmetry to include rotation is momentary stationarity (Bardeen 1970). In time

symmetry, one demands that the spacetime near the t = 0 slice be invariant when $t \to -t$. For momentary stationarity one demands such an invariance when simultaneously both $t \to -t$ and $\phi \to -\phi$ where ϕ is an axial angle (we assume axisymmetry of the spacetime). Thus, if we wanted initial data describing a collapsing axisymmetric star, we could imagine it has just finished expanding outward, was pausing momentarily and then recollapsing. At that pause, the data would be time symmetric if the star were not rotating. In order to handle a rotating star, we would assume its fluid velocity vector u^a had nonzero components only in the ϕ and t directions. Examples of momentarily stationary initial data are provided by t = 0 slices of axisymmetric, stationary spacetimes such as Kerr black holes,[22] rotating disks,[23] or rotating stars.[24] Time dependent spacetimes include collapsing rotating stars,[25] two rotating black holes colliding head-on,[26] and a large class of Bianchi type IX spatially homogeneous universes.[27]

Finally, there is a class of "time antisymmetric" conformally flat slices ($\pi^{ab} \neq 0$, $\beta_{ab} = 0$) first described by Brill (1961) and later emended by York (MTW p. 536). This class of initial data sets has not been explored as to its physical content, although York (private communication 1973) has proposed to use it as an alternate method for obtaining a non-time-symmetric two black hole initial slice.

We see that one can give a very rough classification

of possible initial data by asking whether $\pi^{ab} = 0$ (time symmetric) or $\beta^{ab} = 0$ (conformally flat). One could also ask whether R_{ab} was zero or not. Brill and Deser (1968) note that one will have flat space if both π^{ab} and R^{ab} vanish. Similarly one might use the Weyl tensor components E_{ab} and B_{ab} to classify vacuum initial data. We note that for time symmetry $B_{ab} = 0$ and $E_{ab} = -R_{ab}$. It would be nice to have a comprehensive listing of all known exact initial value problems along with the values of (π^{ab}, β^{ab}, h^{ab}, R^{ab}, E^{ab}, B^{ab}) so one could develop a better intuition as to the physical significance of these tensors in the evolution off the initial surfaces.

References

Chapter IV

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11. Dirac (1959).
12. Misner (1969).
13. Kuchar (1971, 1972).
14. York (1972).

15. York (1971, 1972, 1973). See also a remark in ADM (1962) p. 250.
16. Isenberg (1972) has calculated β_{ab} for a number of gravitational wave solutions.
17. See Smarr and DeWitt (1973).
18. Weber and Wheeler (1957), Brill (1959), Araki (1959), Wheeler (1961, 1964).
19. Misner (1960, 1963), Lindquist (1963), Brill and Lindquist (1963), Wheeler (1961, 1962, 1964).
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21. O'Murchadha and York (1974), see also Brill (1959), Araki (1959), Brill and Deser (1968).
22. Kerr (1963), Carter (1968), Bardeen, Press, and Teukolsky (1972).
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26. Hawking (1973).
27. Matzner, Shepley, Warren (1970).

Chapter V

We now turn from the initial data formulation to its Cauchy development. It is well known from the work of Lichnerowicz[1] and Fourès-Bruhat[2] that if we specify initial data (h_{ab}, K_{ab}) satisfying the constraint equations on a spacelike hypersurface S, then the Einstein evolution equations will give us the unique spacetime in a finite neighborhood of S. The proof relies on treating the Einstein equations either as a second order system of symmetric hyperbolic equations (Bruhat and Lichnerowicz) or more recently as a first order quasilinear symmetric system (Fischer and Marsden 1972).

In a useful article, Choquet-Bruhat and Geroch (1969) explain some global aspects of this evolution. They show that if one has several developments from a given set of initial data then (1) these developments must agree in their intersection and (2) their union is or is contained in a maximal Cauchy development. What we would like to consider is how to slice forward from a given initial data set in such a way that we obtain as large a subset of the maximal development as possible.

This is a very difficult problem in general, because one does not know anything about the spacetime one is evolving until it appears slice by slice. Like jungle explorers we must push forward as quickly as possible (our

computer funds are finite), but we must try to anticipate whether we are about to run into a snarling physical singularity or a treacherous coordinate crossing. Either one will stop the calculation. Further, we have to find a way to sift the physics out of the coordinates and piles of computer output, just as a gold miner has to search carefully for nuggets in rock or sand.

Thus, we are concerned with more than the question of whether a development exists or is unique; we are concerned with how to construct a slicing procedure that in some way tells us when to slow down in one region ($\alpha \to 0$ if we get too close to a singularity) and when to go full steam ahead in another. Since we have to rely on coordinates to label our points, we also want a method which keeps our coordinate lines from crossing one another and thus giving us the appearance of a singularity when in fact spacetime is well behaved.

Since our approach is basically geometric in spirit we will concentrate on the spacelike hypersurfaces and the congruence of normal vectors that they generate. This means we will look at coordinate conditions as restrictions on the choices of α and β_a. There are other methods available to us such as imposing the spacetime harmonic coordinates used in the proofs on Cauchy development mentioned earlier,[3] by using orthogonal coordinates to make the spacetime metric block diagonal,[4] or by using anholonomic frames and the

calculus of forms.[5] These are all areas that need more investigation to see if they are practical methods for constructing developments.

The slicing we obtain depends essentially on what condition we put on α. We can change β_a if we wish (and often this is useful), but this will merely recoordinatize the slices and produce the v^a coordinate line congruence in addition to the n^a congruence of normals. For this reason we will mainly concentrate on α now and mention the possibilities for β_a later. Also, we are concerned with the abstract properties of various slicing now and will give concrete examples at the end of the chapter.

Of course the simplest choice is $\alpha = 1$, $\beta_a = 0$. This is the geodesic normal or Gaussian coordinate system. It is normal because $\beta_a = 0$, geodesic because $\alpha_{,i} = 0$. If we think of the timelike trajectories traced out as world lines of "test dust," then for this case the dust is freely falling. Note that because $\dot{\alpha} = 0$, the time t labeling the hypersurface is also proper time along the geodesics. It is well known that such a system self intersects in a finite time if it is initially normal to hypersurface S with $K > 0$.[6] We can easily see why by considering the Raychaudhuri equation or Einstein evolution equation for K. Since $a_a = 0$, $\alpha = 1$, $\beta_a = 0$, $\omega = 0$ we have:

$$\pounds_u \theta = -\tfrac{1}{3}\theta^2 - 2\sigma^2 - \tfrac{1}{2}(\rho + S) \quad .$$

Now the right hand side is negative definite if the strong energy requirement holds. Thus $\pounds_u\theta < 0$. This means that if the surface is maximal ($K = -\theta = 0$) initially, the normals will begin to converge as soon as any matter is encountered or any shear develops. Since shear here means tracefree extrinsic curvature, we have seen that this will become nonzero (if it is not initially) as soon as some Weyl tensor (E_{ab}) is present. Unless the spacetime is conformally flat or very special ($E_{ab} = 0$ but $B_{ab} \neq 0$) this happens immediately. Once the convergence process begins it accelerates itself since $\theta \neq 0$ just makes $\pounds_u\theta$ bigger. The end result is that $\theta \to -\infty$ or $K \to +\infty$ and the coordinate system self intersects.

The question is: "how finite" is this time before intersection compared to the proper times along other geodesics normal to S initially. In other words, if all these geodesics hit a physical singularity simultaneously, such as is the case for a Friedmann universe starting from the moment of time symmetry, then the geodesic normal coordinates will cover the maximal development, and their focusing will be a focusing of the entire spacetime into a physical singularity. However, if we have an asymptotically flat spacetime, like a black hole, then it is more likely that the geodesics in the strong field region will focus first while the geodesics that started farther out will still be in quite nonsingular regions of spacetime when "the bell tolls"[7] for some central point of S where $\theta \to -\infty$. In this case one does not know

immediately whether the crossing occurs just as the geodesics hit the singularity or whether they go bad first. As we will see, in Schwarzschild the crossing and the physical singularity occur simultaneously (the throat "pinches off"). To determine the answer in an unknown spacetime (such as the two black hole collision) we can keep track of the invariants I and J or watch the scalars on S like $E_{ab}E^{ab}$, $B_{ab}B^{ab}$, $E_{ab}B^{ab}$, etc. to see if they become infinite at the crossing.

In what follows we will assume that a calculation must cease when a coordinate or physical singularity develops on some S_t. This is the Wheeler philosophy that "smoke rises from the computer and we can go no further." We are currently investigating the possibility of simply ignoring the "bad region" and continuing onward assuming that the singularity can not propagate out through our mesh faster than light rays. If it is a real singularity, it will be in a trapped region (see below) and thus will never affect regions that can get to "observers at infinity." Unfortunately, there are a number of elliptic equations in this game, and they can affect events in a spacelike fashion so this is a delicate issue.

The next stage of generalization is to keep $\dot{\alpha} = 0$ but allow α to vary over S, i.e. let $D_c\alpha \neq 0$. For asymptotically flat space we usually would like $\alpha \to 1$ so that we manifestly have Minkowski space there. This means in practice that $0 \leq \alpha \leq 1$ on S. An example of this type of development

is the t = constant slices in Schwarzschild. Again, unless circumstances are very special (and it happens that in stationary spacetime they are) the normals will begin to shear and converge. The Raychaudhuri equation now has an acceleration term ($\alpha_{,i} \neq 0$) and the congruence is no longer geodesic. This term can be of either sign so that the focusing may be faster or slower depending on the particular spatial dependence of α chosen. On the other hand since $\alpha < 1$ over much of S, t is no longer proper time along the normal congruence ($d\tau = \alpha dt$). This means that even if the central region hits a singularity, the asymptotic region will have gone further before we have to stop calculating than it would have if $\alpha = 1$ everywhere. This is particularly crucial if we expect radiation to be generated in strong field regions and then to have time enough to get out to the "wave zone."

Finally, the most general condition is for both $\dot{\alpha}$ and $D_c\alpha$ to be nonzero. There are two main subclasses here: algebraic and differential conditions on α. An example of the former would be a system whereby one monitored some measure of curvature, such as $E_{ab}E^{ab}$ or K, and then made α smaller "by hand" in those regions where the curvature became large. This would effectively slow down the evolution on those portions of S where it seemed that we might be in danger of hitting a singularity. The major example of the differential condition is the <u>maximal slicing condition</u>. This condition (called extremal or minimal by some) obtains when

$K = \dot{K} = 0$ on S_t. As Lichnerowicz pointed out in his 1944 paper, this condition means that the normal congruence behaves like an irrotational incompressible perfect fluid. The coordinate lines may shear, but they do not converge or expand ($\theta = 0 = \dot{\theta}$). We immediately obtain the condition on α by setting $K = \dot{K} = 0$ in the evolution equation for K:

$$\pounds_{\alpha n} K = \alpha K^2 + \alpha R - D^a D_a \alpha + \tfrac{1}{2}\alpha(\rho + S)$$

$$D^a D_a \alpha - R\alpha - \tfrac{1}{2}(\rho + S)\alpha = 0 \qquad \text{(maximal)}$$

This simple elliptic differential equation was derived by Lichnerowicz and has since been discussed by a number of authors.[8]

We will now describe a few of the advantages and disadvantages of the maximal slicing condition. As was first shown by Bompiani (1921), such a slicing is volume-preserving for the normal congruence. Physically that is what we expect if the volume expansion $\theta = 0$. This means that the determinant of the 3-metric is conserved along n^a. The maximal equation is derivable from a variational principle $\delta v = 0$ where v is an integral over the invariant volume element for the slice, fixed on some 2-sphere boundary. The fact that it maximizes volume (because it is a +++ metric embedded in a +++- spacetime) is very important for our purposes. It gives a reason why in a closed universe one has at most one

$K = 0$ slice--the moment of maximum expansion.[9] Furthermore, it indicates that such a slicing will tend to stay away from singularities because for "Schwarzschild-like" $r = 0$ singularities, volume becomes small near the singularity. Thus, we might guess such a slicing makes $\alpha \to 0$ as it feels the curvature mounting and the volume decreasing, while it lets $\alpha \approx 1$ in regions of smaller curvature. In fact, for maximal slices of Schwarzschild we find that this is precisely what happens.

Of course, because the maximal equation is elliptic, we may obtain several slicings depending on the boundary conditions we put on α. However, we note that one choice leads to the natural rest frames in stationary spacetimes (i.e. t = constant in Kerr and Schwarzschild).[10] If we keep this same boundary condition for nonstationary spacetimes we presumably have as good a candidate as any for the "natural" rest system or "natural" system of observers.

Besides these attractions we know from the work of Dirac, ADM, and York that the trace of K_{ab} is essentially a gauge variable. By freezing it out ($K = 0$), we make at least one of our field quantities K_{ab} symmetric, transverse, and traceless. In a calculation of a nonstationary spacetime where radiation is expected to be generated and propagate, it behooves the would-be spacetime explorer to try to get his coordinate variables as close to physics and as far from gauge as possible. Maximal slicing seems an eminently

reasonable way of doing this.

There are two big drawbacks unfortunately. First, as Lichnerowicz pointed out thirty years ago, one does not possess existence theorems for maximal surfaces in spaces of indefinite metric. As far as the author knows, today the situation is not much clearer. It must be that with all the theory of elliptic equations at our disposal one can get some powerful theorems on when families of maximal slices exist in general relativistic spacetimes. As we mentioned before, the boundary conditions imposed will be all important.

Secondly, the maximal slicing condition <u>is</u> differential. This means that it presents real difficulties for time, storage, and accuracy when one is doing 2 + 1 numerical calculations because this elliptic equation will have to be solved on every time step to ensure $\dot{K} = K = 0$. We have shown it to be practical on one space + one time dimensional problems, such as maximally slicing Schwarzschild spacetimes. The problem has been set up for the two black hole problem and a few trial runs made. It looks as if it will be feasible here also, although the cost in time and space (in the computer!) is fairly large.

Finally, we turn to conditions on β_a. One might wonder why one would ever need to use $\beta_a \neq 0$. There are three principal physical reasons: 1) we may wish to push our coordinate lines "radially" outward to prevent them from being sucked into the strong field region, 2) we may try to

prevent the lines from shearing to a singularity, and 3) we may want a rotating or twisting coordinate system which follows matter flow. In fact, these three situations have presented themselves in actual numerical calculations: 1) in the maximal slicing of one black hole,[11] 2) in the non-maximal slicing of two black holes,[12] and 3) in calculations of rotating star collapse.[13] Unfortunately, there is no good idea yet for a differential or algebraic method of choosing β_a in a way that specifically gets rid of one of these three problems. One final condition should be mentioned for completeness--the maximal harmonic slicing.[14] Here one chooses α by demanding a maximal slicing and β_a by demanding harmonic coordinates in the 3-space. The four differential equations (in vacuum) that result are:

$$\alpha_{\cdot i}{}^{i} - R\alpha = 0$$

$$[\gamma^{1/2}(\beta^{i}{}_{\cdot}{}^{j} + \beta^{j}{}_{\cdot}{}^{i} - \gamma^{ij}\beta^{k}{}_{\cdot k}) + 2\alpha\pi^{ij}]_{,j} = 0$$

$$\alpha \to 1 \qquad \beta_i \to 0 \qquad \text{as } r \to \infty$$

This would require solving four elliptic equations on each time step which would probably be overdoing it. One would spend all one's time on the computer on coordinates and very little time on physics!

As mentioned above, maximal slicing keeps K_{ab}

traceless, and it is already transverse from the momentum constraints. It would be nice to find a way to propagate just a symmetric, transverse, traceless tensor which represented the metric. Deser (1967) investigated equations for K_{ij} and R_{ij}, and one can also find equations for K_{ij} and β_{ij}. Unfortunately, these also involve the γ_{ij}, which would also have to be evolved. The point is that one would like to build a formalism in which only the "spin-2" parts of the field appear. This would be a technique analogous to the notion of superspace,[15] where each point represents a three geometry, and a particular slicing gives a particular path through superspace. As York (1972) points out, it is more physical to speak of conformal superspace where each point represents a conformal 3-geometry (γ_{ij} modulo diffeomorphisms modulo conformal transformations), and a path represents the evolution of the "spin-2" gravitational field.

Before moving on to particular examples we briefly review some of the important physical processes we expect to find in highly curved nonstationary spacetimes. As Thorne (1972) points out, the three phases of a generic collapse will be implosion, horizon, and singularity. Whether we collide black holes or collapse stars this is the basic scenerio. One expects copious emission of gravitational radiation as well, but this is the one aspect most difficult to get a quantitative handle on.

The implosion should be fairly easy to follow on

the computer since much of it will be in weak field regions. If matter is present, one may find shock waves develop, but there are battle-tested techniques for dealing with these effects. As the matter becomes compacted into a smaller and smaller volume, a trapped surface[16] may form. This is a closed 2-surface with outgoing null normals of which the divergence is zero or negative. Penrose (1965) introduced this notion as an indicator that one was in a region where the curvature of spacetime was becoming very important. It can also be used to show that a singularity must develop in the future,[17] assuming causality does not break down and that the energy conditions continue to hold.

The nature of this singularity has been the subject of much study. As we will see shortly, the Schwarzschild singularity which is spacelike and hidden behind an event horizon is not stable for vacuum solutions. If one adds the tiniest amount of electric charge (producing Reissner-Nördstrøm) or angular momentum (producing Kerr), then the singularity becomes timelike.[18] Therefore, it is of considerable interest to find out the types of singularities generated by highly nonspherical collapse or collisions of black holes.

The more important physical question is whether the singularity, whatever type forms, is visible from infinity. This is the issue of Cosmic Censorship[19] raised by Penrose. If causal curves from arbitrarily near

singularities can get far away (i.e. to Earth), then, as Hawking stresses,[20] our power to predict the future will be destroyed because one will no longer have a nonsingular Cauchy surface. Anything could and might "pop out" of the singularity and influence physics. Penrose has actually suggested that such a naked singularity might provide a source of powerful gravitational radiation. Now we know that this does not happen for spherical collapse or even for perturbations from spherical collapse,[21] but it is an entirely open question as to whether the Cosmic Censor acts in highly nonspherical collapse. In fact, a number of recent investigations have indicated that in certain "more or less" physical collapse scenerios,[22] naked singularities do form. Therefore, it is imperative to investigate what happens in realistic collapse problems. Here again the computer will be our chief tool.

The cloth that the Censor uses to hide the naked singularity is technically referred to as an event horizon. It is the boundary of the set of events in the spacetime which can not be connected by causal curves to null infinity. As such it is a null hypersurface with null generators possessing no future endpoints, but which may have past endpoints. As our spacelike slices S_t progress in time this null hypersurface intersects the S_t in spacelike 2-surfaces which are called the <u>surfaces of black holes</u>.[23] These 2-surfaces may merge into larger 2-surfaces as we go from one

S_t to another. It is a fundamental law of black holes that the total surface area of all black holes on a S_t is less than or equal to the total surface area on S_{t+dt}. That is, the area of black hole surfaces always increases. Further one can show that black holes can never bifurcate. All these theorems (proved by Hawking) depend on our assumption of Cosmic Censorship. Thus, if one finds an example where areas decrease, we will know naked sigularities can exist.

Again the computer will give us more insight into the nature of horizons. Already May and White[24] have shown that one can follow null rays on the computer for star collapse and Cadez[25] has done the same for black hole collisions. Hopefully, we can someday produce a movie showing the behavior of null rays in such a collision.

There is a local measure of the approximate location of the horizon and that is the apparent horizon introduced by Hawking (1973). This is the boundary of trapped surfaces in a given S_t. It is clear that in a dynamic situation the event horizon lies outside the apparent horizon, because the null generators of the event horizon have positive divergence then, while the outgoing null rays orthogonal to the apparent horizon have zero divergence. This "gap" narrows as the physical situation becomes stationary, and in the final state (Kerr) one finds the event horizon and apparent horizon coincide. Now the great advantage of the apparent horizon is that it can be found on each S_t by

searching for a surface whose outgoing null normals have zero divergence. Cadez (1974) has shown this is practical on the computer by finding the apparent horizons for the time symmetric two black hole initial slice. There he illustrates numerically what Brill and Lindquist (1963) had earlier discovered and what Hawking has abstractly considered, namely that the apparent horizon is not a continuous surface, but may suddenly jump from being two surfaces, one around each hole, to being a single surface surrounding both holes. The topology of connected components of the apparent horizon is always that of a 2-sphere.[26] In contrast, the event horizons of two black holes first deform, then touch, and finally merge in much the fashion that a fissioning nucleus appears if we watch it with time running backward.

In this process, as in the process of nonspherical collapse, one expects large amounts of gravitational radiation to be generated. Now gravitational radiation as such is not well defined in a strong field region. One has highly curved spacetime with "Waves, Newtonian Fields, and Coordinate Functions"[27] all mixed up. It is only when we move farther away from the central region that the curvature sorts itself out into large scale variations due to the central mass and short wavelength wiggles on this curved background. Isaacson (1968) has shown how to construct an effective stress tensor for this radiation using an averaging technique developed by Brill and Hartle.[28] More recently, O'Murchadha

and York (1974) have shown how to make a covariant decomposition of perturbations, a generalization of the ADM approach which first rigorously defined radiation for us in the spacelike slicing approach.[29]

Thus, on each slice S_t we will have a chaotic strong field region, a region of outgoing radiation, a wave front, and then quiescent space. Our problem is to define a method for use in 2 + 1 computer codes that will take the given (h_{ab}, K_{ab}), put it into a subroutine, and calculate on it, so that what comes out will be a radiation pattern with the energy content of the waves. We have not yet devised such a program, but our research is continuing.

Several approaches besides the Isaacson method might be considered. Since gravitational waves will follow null rays as they leave the strong field region, we could compute null surfaces from our "stack" of spacelike surfaces and then use the powerful techniques developed by Pirani, Sachs, Penrose, Newman and others[30] to calculate the radiation. Alternatively, we could monitor the dynamical components Q_{ab} of the Riemann tensor. These contain the components $R_{t\theta t\theta}$ and $R_{t\theta t\phi}$ which enter the Newman-Penrose quantity Ψ_0 in the far zone.[31] One might use them in a Landau-Lifshitz pseudotensor to calculate the power of the outgoing waves.

Finally, there are more speculative approaches. It would be nice if one could derive the radiation from $\tilde{\beta}^{ab}$ and $\tilde{\pi}^{ab}$ alone. They seem to be the objects most representative

of the spin-2 nature of the gravitational field. Isenberg (1972) has shown that for some exact gravitational waves, π^{ab} and β^{ab} take very simple forms which clearly show the polarization of the waves. There should be more research done on the Bel-Robinson tensor[32] to see if it can give us some insight into the nature of the radiation. Finally, one might go back to some of the notions Dirac (1959) had for splitting the Hamiltonian into parts representing Newtonian fields and parts representing gravitational waves, to see if they can be applied to our situation.

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Chapter V

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4. Rosen and Shamir (1957), Bonner (1959), Chandrasekhar and Friedman (1972).
5. Bardeen (1960).
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8. Lichnerowicz (1944), Komar (1963), Misner (1963), Peres (1960), Brill (1964), DeWitt (1967), Cadez (1971), Estabrook, et al. (1973).
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13. Pachner (1973).
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15. Wheeler (1964, 1968), DeWitt (1967), Fischer (1970), DeWitt (1970), Ebin (1970), Misner (1972).
16. Penrose (1965, 1968), Hawking (1973), Hawking and Ellis (1973).

17. Hawking and Penrose (1969).

18. See for instance Liang (1973).

19. Penrose (1969).

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22. Thorne (1972), Yodzis, et al. (1973), Liang (1973).

23. Hawking (1973). See Smarr (1973) for discussion of the the surface of a Kerr black hole.

24. May and White (1966).

25. Cadez (1971).

26. Hawking (1973).

27. Title of article by Misner (1964).

28. Brill and Hartle (1964), Brill (1964), MTW (1973), DeWitt--lecture notes (1971).

29. ADM--Witten volume (1962), Misner (1964).

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31. Price and Sandberg (1973).

32. See for instance the discussion in Penrose (1960).

Chapter VI

We will now apply the tools we have built up to the study of particular spacetimes. First, we give an outline of the various combinations of congruences we might encounter. Then we look in detail at the slicing of one black hole. For the nonrotating case we compare geodesic normal slicings with various maximal slicings and show how shift vectors may be introduced. Twisting congruences are naturally examined in the Kerr spacetime. For examples of matter congruences we turn to the spatially homogeneous cosmologies. Finally, we review the types of slicings used in some previous numerical calculations of gravitational collapse.

As mentioned in Chapter III, we may have as many as three natural congruences associated with a slicing of spacetime. The slicing itself defines a congruence of normal vectors n^a. If we use a nonzero shift vector, the coordinate line congruence v^a will also be present. If matter is present its flow lines w^a may or may not lie along n^a or v^a. For each of these congruences, generically labeled u^a, we can examine the acceleration, expansion, shear, and twist. Some formulae relating these quantities for the n^a and v^a congruences were given at the end of Chapter III.

A congruence u^a which does not lie along the normal congruence n^a will be called a tilted or nonnormal congruence.

If w^a lies along v^a we will say we are in a comoving coordinate system. If furthermore the time coordinate t is the same as the proper time along the matter congruence we say we are in a synchronous comoving coordinate system. In this last case, one is effectively using the matter flow lines to define coordinate lines. We stress that it is up to the "slicer" to decide which of these techniques for slicing he cares to use in a given problem.

We can give a breakdown of the various possibilities as follows:

Vacuum - n^a and v^a

I. Normal Coordinates - $\beta = 0$, $v^a = n^a$

II. Nonnormal Coordinates - $\beta \neq 0$, $v^a \neq n^a$

Nonvacuum - n^a, v^a, w^a

I. Normal Coordinates - $\beta = 0$, $v^a = n^a$

A. Comoving Matter - $w^a \parallel n^a$

1. Synchronous $t = \tau$

2. Nonsynchronous $t \neq \tau$

B. Tilted Matter - $w^a \nparallel n^a$

II. Nonnormal Coordinates - $\beta \neq 0$, $v^a \neq n^a$

A. Normal Matter - $w^a \parallel n^a$, $w^a \nparallel v^a$

B. Comoving Tilted Matter - $w^a \nparallel n^a$, $w^a \parallel v^a$

1. Synchronous $t = \tau$

2. Nonsynchronous $t \neq \tau$

C. Noncomoving, Nontitled Matter - $w^a \nparallel n^a$, $w^a \nparallel v^a$

Representative examples of some of these cases are for vacuum: I, Schwarzschild and II, Kerr in their usual coordinate systems; for nonvacuum: IA1, May and White collapse or Friedmann universe, IB, tilted Bianchi IX cosmologies, IIB1, Pachner's rotating collapse. Now we will show shortly that one spacetime can be sliced in several of these subclasses, i.e. Schwarzschild is vacuum, but it can be sliced as I or II.

We also recall some terminology regarding the nature of the congruence: $a_b = 0 \Rightarrow$ geodesic, $w_a = 0 \Rightarrow$ hypersurface orthogonal, $w_a \neq 0 \Rightarrow$ twisting, $\theta_{ab} = 0 \Rightarrow$ rigid, $\sigma_{ab} = 0 \Rightarrow$ shearfree. For n^a, if $\theta = 0$ we have a maximal slicing. Notice that $w_a = 0$ does not mean the congruence u^a must be orthogonal to the slicing hypersurfaces. For instance, one might use a nontwisting shift vector to produce a v^a congruence. This consequence is <u>not</u> orthogonal to the S_t but there does exist a family of surfaces to which it is orthogonal. In this special case one could interchange the roles of v^a and n^a and thus obtain two distinct slicings of the same spacetime.

Our first example will be the Schwarzschild spacetime. As usual the very high degree of symmetry of this solution will hide a number of generic properties of slicings. However, we will be able to illustrate concretely a number of the points we have been discussing. It will also serve as the prototype for the two black hole spacetime we cover in

the next chapter.

The vacuum Schwarzschild solution differs from the solution of a spherical star with vacuum exterior in that the topology of the t = constant slices are noneuclidean in the former. That is, the manifold we use to solve the initial value problem on for vacuum has the topology of an Einstein-Rosen bridge[1] connecting two asymptotically flat sheets. For a star, one has only a deformed or stretched sheet with a pocket of matter at the bottom.[2]

For the initial slice we will assume time symmetry ($K_{ab} = 0$) and conformal flatness ($\beta_{ab} = 0$). The momentum constraints being identically satisfied, we have left only the scale equation which reads:

$$\Delta\phi = 0$$

where Δ is the Laplacian in flat space coordinates. We want a spherically symmetric solution which has asymptotic behavior:

$$\phi \to 1 + \frac{M}{2r} \qquad r \to \infty \quad .$$

This is satisfied only if $\phi = 1 + \frac{M}{2r}$ everywhere. Now $r = 0$ has nothing whatsoever to do with the "point" singularity at the origin of the spherical symmetry. That is because we are using isotropic coordinates where $r \to 0$ corresponds to $r \to \infty$ on the lower sheet. The existence of a mirror symmetry

through the throat $r = \frac{M}{2}$ means that we can change coordinates to $r' = \frac{M^2}{4r}$ so that

$$ds^2 = (1 + \frac{M}{2r})^4 (dr^2 + r^2 d\Omega^2) = (1 + \frac{M}{2r'})^4 (dr'^2 + r'^2 d\Omega^2)$$

and thus if $r \to 0$, we can switch to $r' \to \infty$ and the metric is seen to be regular. In any case, we shall see shortly that the central singularity is not a point but a 3-surface hanging "above" the initial sheet.

We note that the throat $r = \frac{M}{2}$ is a compact minimal 2-surface in a time symmetric 3-surface. This means it is a marginally trapped surface.[3] We will see from the full spacetime that it is the apparent horizon for the initial sheet and that it also lies on the event horizon. This is because the spacetime is stationary and therefore the null generators of the event horizon are nonexpanding. As mentioned before in this case, the apparent horizon coincides with the event horizon.

We can evaluate the curvature invariants I and J on the initial sheet. We have $B_{ab} = 0$ because $K_{ab} = 0$, and since there is no matter present, we have $E_{ab} = -R_{ab}$. Thus, we simply obtain:

$$I = 4R_{ab}R^{ab} \qquad J = -8R_{ab}R_c{}^b R^{ca}$$

We may evaluate these in any coordinate system by calculating

R_{ab}. There are four sets of coordinates we will find interesting:

Isotropic $\quad ds^2 = (1 + \frac{M}{2\rho})^4 (d\rho^2 + \rho^2 d\Omega^2)$

$$I = \frac{48M^2}{\rho^6 (1 + \frac{M}{2\rho})^{12}}$$

Schwarzschild $\quad ds^2 = (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\Omega^2$

$$I = \frac{48M^2}{r^6}$$

Kruskal $\quad ds^2 = \frac{32m^3}{r} e^{-r/2m} du^2 + r^2 d\Omega^2 \qquad r = r(u)$

$$I = \frac{48M^2}{r(u)^6}$$

Conformal $\quad ds^2 = 16a^2 \cosh^4(\frac{\eta}{2}) (d\eta^2 + d\Omega^2)$

$$I = \frac{3}{64} a^{-4} \cosh^{-12}(\frac{\eta}{2})$$

This shows how different initial metrics can look while still representing the same 3-geometry. Note that the middle two do not appear to be conformally flat at first glance. However, if one calculates R_{abc} for each of the four, one will get zero in every case. The last of the four is written in terms of dimensionless coordinates μ,θ,ϕ. We will see in the next chapter that these coordinates are analogous to those

Misner used to pose the initial conditions for two black holes.

To obtain a development from this initial slice we have to choose coordinate conditions on α and β_a. Fortunately we know the maximal development of this slice[4] and can graphically represent it by the familiar Kruskal diagram. This is not the situation in general. Usually one does not have the spacetime a priori, but must find it slice by slice. The usefulness of studying Schwarzschild is that it allows us to "calibrate" our intuition by comparing various slicing conditions in a situation where the global setting is manifest.

The first coordinate system we investigate is the Gaussian or geodesic normal system. This requires us to set $\alpha = 1$, $\beta_a = 0$. Since the radial geodesics of Schwarzschild are known we can calculate the congruence exactly. We have for initial conditions a position $R = r_0/2M$ and a velocity zero. Here r_0 is the usual Schwarzschild coordinate on the $t = 0$ initial slice. The resulting position in Schwarzschild coordinates (r,t) after proper time τ has elapsed can be given parametrically (see MTW) by:

$$\tau = MR^{\frac{3}{2}}(\eta + \sin\eta)$$

$$r = MR(1 + \cos\eta)$$

$$t/2M = \ell n\left(\frac{(R-1)^{\frac{1}{2}} + \tan(\eta/2)}{(R-1)^{\frac{1}{2}} - \tan(\eta/2)}\right) + (R-1)^{\frac{1}{2}}[\eta + \tfrac{1}{2}R(\eta + \sin\eta)]$$

The values of (r,t) can be converted into position on the Kruskal diagram by using the transformations to Kruskal coordinates (u,v):

$$u = (r/2M - 1)^{\frac{1}{2}} e^{r/4M} \cosh(t/4M) \qquad r > 2M$$

$$v = (r/2M - 1)^{\frac{1}{2}} e^{r/4M} \sinh(t/4M)$$

$$u = (1 - r/2M)^{\frac{1}{2}} e^{r/4M} \sinh(t/4M) \qquad r < 2M$$

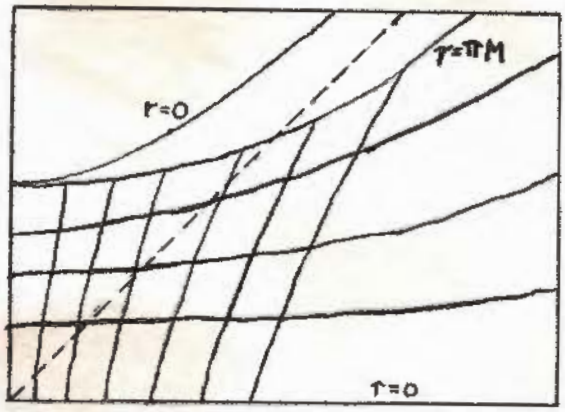
$$v = (1 - r/2M)^{\frac{1}{2}} e^{r/4M} \cosh(t/4M)$$

Using the formulae we map out the τ = constant hypersurfaces and the orthogonal geodesic congruence of the spacetime metric:

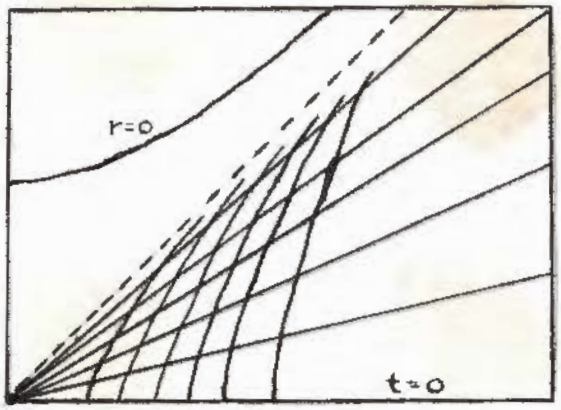
$$ds^2 = -d\tau^2 + A(R,\tau)dr^2 + r^2 d\Omega^2 \quad ,$$

where $r = r(R,\tau)$. The result is shown in Fig. 1.

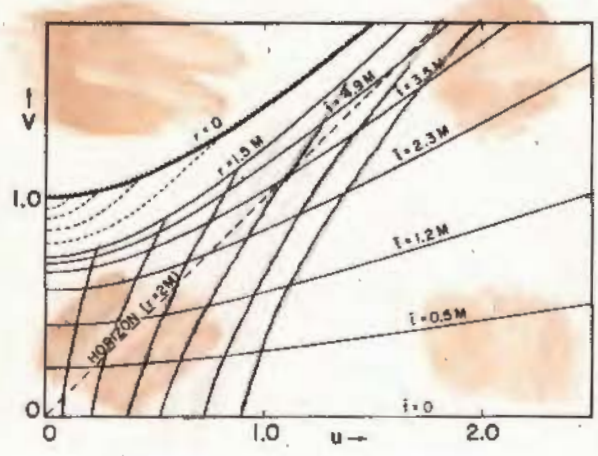
We note that the slicing becomes singular on hypersurface $\tau = \pi M$. Since the τ = constant surfaces are asymptotically Schwarzschild t = constant surfaces, this means we can evolve only about 3.14 M before we stop computing. Clearly if this had been a problem in which the black hole was to generate radiation, we would not have had enough time for the radiation even to begin to propagate outward to our t = constant "observers at infinity," since light travel times for the system are on the order of 2M.



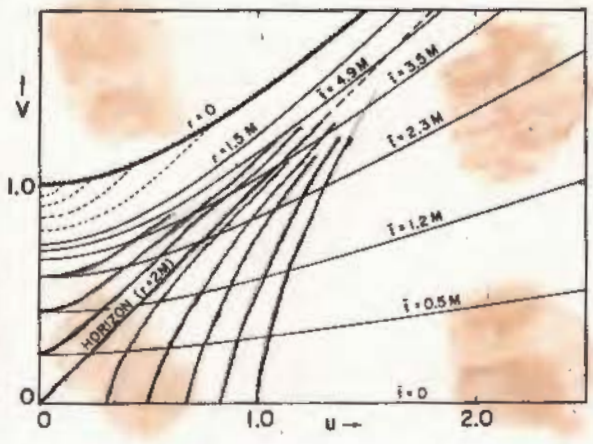
Geodesic Normal



Maximal $\dot{\alpha} = 0$



Maximal $\dot{\alpha}\neq 0$ $\beta_r = 0$



Maximal $\dot{\alpha}\neq 0$ $\beta_r \neq 0$

Figure 1

On the other hand we did penetrate deep "beneath" the surface of the black hole and we were able to sample regions arbitrarily close to the singularity. If the horizon ($r = 2M$) were evolving, our spacelike hypersurfaces would have intersected its null surface, and we could have recorded the details. One further feature is that the coordinate lines near the hole fall behind the horizon as time goes on.

Our next slicing will be a maximal one. We must have boundary conditions on α if we are to obtain a unique solution of the lapse equation:

$$\Delta\alpha - R\alpha = 0 \quad .$$

Because of the noneuclidean topology we have two boundaries: the usual one at infinity where $\alpha \to 1$ and a new one at the throat ($u = 0$ in Kruskal). Because of the symmetry of Kruskal about $u = 0$ we have only two choices. The first is $\alpha(-u) = -\alpha(u)$ which implies that $\alpha = 0$ on the throat. The second is $\alpha(-u) = \alpha(u)$, which leaves α free but requires the slice to strike the line $u = 0$ perpendicularly.

The first slicing yields the Schwarzschild $t =$ constant slices. We can see that as follows. Since on the $t = 0$ slice the Hamiltonian constraint implies $R = 0$ we have:

$$\Delta\alpha = 0$$

where the Laplacian is with respect to the full metric $ds^2 = \psi^4(d\rho^2 + \rho^2 d\Omega^2)$. The spherically symmetric solution which has $\alpha \to 0$ as $\rho \to M/2$ is:

$$\alpha\psi = 1 - \frac{M}{2\rho}$$

Then one can show that ${}^{\alpha}R_{ab} = D_a D_b \alpha$ on the initial surface. This means $\pounds_{\alpha n} K_{ab} = 0$ so $K_{ab} = 0$ on each succeeding slice. Thus, $\pounds_{\alpha n} h_{ab} = 0$ and each slice is isometric to the first. However, these are just the properties of the Schwarzschild t = constant slices. The normal congruence is rigid and accelerating because the coordinate lines are the Schwarzschild r = constant trajectories of the time killing vector $\partial/\partial t$. The slicing covers the entire exterior spacetime of the black hole, but it does not penetrate the horizon. This indicates such a slicing will be good for radiation study, but useless for studying horizons. The last point is not correct though since in a time dependent problem we will tie down $\alpha = 0$ at the minimal surface, which will be inside the horizon, and thus we will be able to see the event horizon evolve.

The last slicing will be maximal but with the other boundary condition on α. Here we let $\alpha = 1$ initially and so ${}^{\alpha}R_{ab} \neq D_a D_b \alpha = 0$ on the first slice. This provides a driving term for $\pounds_{\alpha n} K_{ab}$, and we do get a nontrivial evolution. The normal congruence is shearing and accelerating. At first the slices jump forward fairly uniformly ($\alpha \simeq 1$), but then they

abruptly slow down ($\alpha \to 0$) in the central region. This "braking action" is due to the effect mentioned earlier that maximal slices tend to stay away from $r \simeq 0$ regions. As the central region's evolution slows to zero, the outer region continues more or less like the Schwarzschild t = constant slicing.

We see again the effect of the coordinate lines being "sucked in" to the black hole. We can get a slicing which has the same hypersurfaces as this last one, but with coordinate lines like the t = constant maximal slicing if we use a nonzero shift vector. In particular, as shown in Estabrook, et al. (1973), if we choose a β_a so that our coordinate lines follow the Schwarzschild r = constant hyperboles, then we achieve the compromise sought. This is an example of how the acceleration of the coordinate lines is at our disposal. We can let the acceleration be determined solely by our choice of α (through the maximal slicing equation), or we can add an arbitrary "radially outward" push to the coordinate lines. We stress that the stack of hypersurfaces is not effected by this choice of β_a. The v^a congruence is not twisting as can be seen by evaluating the expression for ω_{ij} given in Chapter III.

The maximal slicing with $\alpha \neq 0$ on the throat gives an explicit example of how one can probe very strong field regions of a black hole without having the throat "pinch-off" before the gravitational physics in the "wave zone" has

had a chance to operate. It also gives us hope that the "automatic singularity avoidance" property of maximal slicing will be a generic property. Furthermore, it shows that at least for 1 + 1 spacetime evolution problems the elliptic lapse equation and the hyperbolic propagation equations can be solved simultaneously on the computer.

We have now a detailed example of some of the abstract notions introduced in our discussion of the Cauchy problem. Figure 1 shows three different developments of the same initial data. Their overlap is their common development. The maximal slicing with $\alpha \neq 0$ on the throat provides an extension of the t = constant slicing. Finally, one notes that none of these coordinate conditions yields the maximal development. It is an open question as to whether one can find an explicit coordinate condition which will yield such a slicing.

Now we turn to a spacetime for which the maximal development does not give the full manifold--the Kerr solution.[5] Just as in the Reissner-Nördstrøm solution,[6] we have a Cauchy horizon. It is thought that if such black holes are formed from collapse, the matter may destroy the Cauchy horizon and completely alter the underlying singularity structure. Until fully dynamical nonspherical collapse problems are pursued, it will be hard to say what physically occurs.[7]

Again the topology of the t = 0 slice is that of an

Einstein-Rosen bridge with a throat[8] at $r = r_+ = m \pm \sqrt{m^2-a^2}$. The throat is a minimal surface which is on both the apparent and event horizon. The initial slice is momentarily stationary with $K_{ab} \neq 0$[9] but $K^a{}_a = 0$. As far as I know, the conformal class of the three geometry has not been determined although the calculation is straightforward. I would suspect it to be conformally flat. Only the maximal t = constant slices have been determined for Kerr. The coordinate congruence v^a is twisting with $\beta_a \neq 0$. The normal congruence n^a is generated by the locally nonrotating or ZAMO world lines.[10] Notice that while both of these congruences are shearing, neither has expansion or convergence ($\theta = \tilde{\theta} = 0$). This is because of the presence of the two Killing vectors. Further, inside the ergosphere the coordinate congruence actually becomes spacelike even though the ZAMO's are still timelike, and thus our t = constant surfaces are still spacelike. It is clear that calculating slicing of rotating collapse has many subtle features which are absent from the spherical case.

To give an example of a spacetime with matter we turn to the Bianchi type IX universes,[11] which are generalizations of the familiar Friedmann universe. The universes possessa a geometrically preferred set of slices S_t: the slices of spatial homogeneity. The normal congruence is geodesic, shearing, and has nonzero θ except at the moment of maximum expansion. Because of the homogeneity

$D_a K = 0$ on each slice. The matter congruence w^a can be twisting, accelerating, shearing, and expanding. As mentioned before, this class of spacetimes provides examples of maximal initial slices which are time symmetric, momentarily stationary, or general. The 3-geometry may be conformally flat (Friedmann) or not. If not, one finds that $\tilde{\beta}_{ab}$ measures the anisotropy of space.[12] It is practical to carry out numerical calculations of the evolution from all of these initial data (Matzner, Shepley, Warren, 1970).

Turning now from closed universes to asymptotically flat spacetimes we can consider the calculation of the collapse of a rotating star. Pachner[13] has made some preliminary efforts to set up the computer code for such a situation. In his formalism for collapsing dust he takes nonnormal ($\beta_\phi = \beta_\phi(r)$) synchronous comoving coordinates ($v^a \parallel w^a$). The hypersurfaces themselves have geodesic normals ($\alpha = 1$), so from our experience with Schwarzschild it is doubtful that the asymptotic region will have time enough to evolve for radiation to appear before the central region hits a singularity. Note that for rotating collapse one does not know what the spacetime exterior to the star will be (it is Schwarzschild for the nonrotating case). It definitely will not be Kerr until very late stages of collapse.

Finally we review the work on spherical collapse. The first calculations were by Oppenheimer and Snyder (1939).

They studied the collapse of a ball of dust. This is done in comoving synchronous normal coordinates which are geodesic and converging. The extension to include pressure[14] was carried out by May and White (1966, 1967). The nonzero pressure means that the comoving coordinates are no longer geodesic ($\alpha \neq 1$). The Euler equation tells us that:

$$\partial_r \ell n \alpha = -(\varepsilon+p)\partial_r p \quad .$$

In addition, $\partial_t \alpha \neq 0$ (see May and White, 1966). So here we have an example of an α which varies in space and time but for which we do not obtain maximal slices. The choice of time coordinate is made by the decision to follow the fluid lines. It would be interesting to obtain a maximal slicing of this same spacetime for comparison and as a prelude to nonspherical, nonrotating, 2 + 1 numerical calculations.[15] We note in closing that the May and White spacetime gives us a situation where we can compare trapped surface regions and the event horizon in a nontrivial time dependent manner.[16]

References

Chapter VI

1. Einstein and Rosen (1935), Wheeler (1961, 1962, 1964), Fuller and Wheeler (1962).
2. See for example the picture in Thorne (1967).
3. Brill and Lindquist (1963), Hawking (1973), Čadež (1974).
4. See example 4 in Geroch (1970).
5. Kerr (1963), Boyer and Lindquist (1967), Carter (1968, 1973).
6. Example 5 in Geroch (1970). For Kerr see Penrose (1968) or Hawking and Ellis (1973).
7. For a discussion of these problems and the equations necessary for a spherical collapse of a charged ball or shell with nonzero pressure see Bekenstein (1971).
8. For slow rotation see pictures in Brill and Cohen (1966). Otherwise see Carter (1973).
9. For the exact form see Bardeen (1970) and Hawking (1973).
10. Bardeen (1970, 1973), Synge (1960), Bardeen, Press, and Teukolsky (1972).
11. See for instance Matzner, Shepley, and Warren (1970) and references cited therein. For other Bianchi types c.f. Collins and Hawking (1973) or King and Ellis (1973).
12. York (1972).
13. Pachner (1973).
14. See also Misner and Sharp (1964, 1965), Hernandez and

Misner (1966), Misner (1969), and MTW (1973).

15. Ostriker and Smarr will be studying such calculations.

16. May and White (1966), Misner (1969), Smarr (1972, unpublished).

Chapter VII

In these last two chapters we shall present a numerical illustration of many of the aspects of general relativity that we have studied in the earlier chapters. This chapter will deal with the initial data and give a sketch of the expected evolution. The next chapter will detail our numerical approach to the problem.

Almost since the start of general relativity, the two-body problem has been a central issue. Early efforts were made by Bach, Weyl, and Levi-Civita to obtain a static axisymmetric two-body solution,[1] but such efforts were doomed to failure. Any two-body problem must be time dependent unless there are "singularity struts" present or counterbalancing electromagnetic forces.[2]

Since an exact time dependent solution seems unlikely to be produced, the study of two bodies has mainly been confined to an initial data sheet. Three pioneering approaches to this problem emerged in the thirties: 1) the geometrodynamical model of two masses as noneuclidean bridges between asymptotically flat spaces was formulated by Einstein and Rosen (1935), 2) the time-symmetric conformally-flat initial data with real masses on a Euclidean space was set up by Lichnerowicz (1939, 1944), and 3) the EIH approximation method for deriving the equations of motion from the field equations was investigated.[3]

We will not concern ourselves here with the third alternative, because we are not interested in approximations, but rather in finding a full spacetime representing the movement of two bodies. The second method has lain dormant[4] but is currently being revived for studies of nonspherical collapse.[5] The rest of our story follows the approach of Einstein and Rosen, brought to fruition by Wheeler, Misner, and their coworkers in the early sixties.

In 1960 Misner formulated the first analytic set of initial data for the vacuum two body problem. It was for equal masses and contained two free parameters which set the total mass and initial separation distance of the two throats. Because of the particular form for the conformal factor that he chose (the data is time symmetric and conformally flat), the two lower sheets can be "welded" together. If one identifies the two throats, the famous wormhole connecting different portions of the same space is produced.

Brill and Lindquist (BL-1963, L-1963) followed this up by producing a time-symmetric conformally flat initial data set for N throats in vacuum (BL) or with electric fields present (L). They showed, as Lichnerowicz had shown for the "matter" N body problem, that one recovers the Newtonian potential description if the bodies are far apart. For a review of these studies see Wheeler (1961, 1962).

The first attempt at obtaining a numerical solution for the evolution of the initial data was made by Hahn

and Lindquist (1964). Their work demonstrated the feasibility of this approach and confirmed that each throat behaved much like a single Schwarzschild throat. Unfortunately, they were hampered by a number of problems beyond their control. First, the physical setting was unknown. The notion of black holes and horizons, the area theorems, and the scope of the problem had not yet been defined. Second, the Misner coordinates, while useful for obtaining the initial data, proved disastrous for computer calculations of the far field region. Third, these difficulties combined with their use of geodesic normal coordinates meant that almost all their attention was concentrated on questions such as, "how fast do the throats shrink?" In retrospect, it was truly pioneering work--perhaps an effort ahead of its time.

The problem remained in this state until the excitement generated by the idea of black holes and the possible detection of gravitational radiation by Weber, brought it to the forefront of research once again. In the late sixties Bryce DeWitt started a research effort on the feasibility of colliding two black holes on a computer. This led directly to Cadez's and my theses on the problem. Meanwhile Hawking had produced his area theorems which predicted an upper limit of 29% for the efficiency of conversion of rest mass-energy to gravitational radiation by the infall from infinity of two nonrotating, uncharged black holes.

Since the evolution of Misner's initial data, for

certain values of the parameters, would correspond to an example of the collision of two black holes, in the case where the two holes were a bound system, it was of interest to obtain limits on the efficiency of radiation from such systems. Gibbons and Schutz (1972) and Gibbons (1972) used Hawking's notion of apparent horizon to do just that. Meanwhile Hawking (1973) had considerably clarified the evolution of the horizons in such a collision.

With the physical framework fairly well understood it remained to bbtain the actual numerical solution. Cadez made great progress by finding a new coordinate system which was well behaved on the holes and in the far region. Its only drawback was a coordinate singularity at the point midway between the holes. By the use of coordinate transformations he was able to deal with this point as well. He set up the Einstein equations as formulated in Chapter II and demonstrated the feasibility of tracking null geodesics in the evolving spacetime to follow the development of trapped surfaces. However, as with Hahn and Lindquist, he was not able to carry the evolution far enough into the future to see any large change in the initial data. Since he had no graphic display capabilities, he was also hampered in interpreting the computer output. One other important contribution Cadez made was to determine when the Misner initial data represented two separate black holes and when they represented an "already merged" single black hole. From this we have a

better idea of which values for Misner's parameters to choose.

My work consisted of programming codes to test out various ideas DeWitt and I had for coordinate conditions. Further, we treated the Einstein equations as a second order system instead of a first order system. We tried several methods for dealing with the singular point between the two holes (the "saddle point"). I also developed graphic displays[6] to enable us to follow the evolution. My programs enabled us to get an evolution to the point where our coordinate lines focused to a crossing (nonmaximal slicing). This may be a physical singularity like the r = 0 singularity of Schwarzschild. Codes were also developed for maximal slicing, and they are being tested now. Finally, I investigated the possible modes of radiation and the astrophysical setting for colliding black holes. Much of this thesis is also devoted to the possibility of extending these ideas to other problems such as nonspherical collapse of stars.

Before coming to the details of our approach, I should mention that there is another approach to the initial value problem and Einstein's evolution equations. This is the Regge calculus invented by Regge (1961) and pursued by Wheeler (1964) and his coworkers. The initial value problem has been set up for Schwarzschild and Reisnner-Nordstrøm (Wong, 1971), two black holes, and the Friedmann universe (Collins and Williams, 1972). The evolution of the Friedmann

universe suggested by Wheeler (1964) was computed by Collins and Williams in 1973. There is probably much more work that could be done here (see MTW, 1973), but we will not pursue it further.

We now turn to a study of the Misner initial data. As noted before, it is time symmetric ($K_{ab} = 0$) and conformally flat ($\beta_{ab} = 0$). We choose the form of the conformal factor used by Misner rather than that used by Brill and Lindquist because we want there to be an isometry between the upper and lower sheets with a mirror reflection plane through the throats. This is for our computing convenience, not for any physical reason. In fact, one may object to our use of such an idealized initial data set. We will argue later that we expect most of the radiation to be generated by the coalescence of the horizons, and as such we do not expect the resulting evolution to depend very much on choice of initial conditions. As outlined before, we are developing new initial sets which are neither time symmetric, nor conformally flat. If this program is successful we will be able to run the evolution from these initial data and thus directly check the sensitivity of the evolution to changes in initial conditions.

Rather than just write down the Misner matched initial metric, we will derive both it and Cadez's metric by using Cadez's conformal mapping technique. The idea is to write down the conformally flat line element in cylindrical coordinates since they are the natural coordinates for an

axisymmetric system. Then we perform a complex conformal map from (z,ρ) to new coordinates (η,ξ). This is analogous to what we would do if we wrote Schwarzschild down in cylindrical coordinates and then transformed to spherical coordinates by the map:

$$\eta + i\xi = \chi(z + i\rho) = \ell n\zeta$$

where χ is a complex analytic function of $\zeta = z + i\rho$. The logarithm function is used because it puts a pole at the origin. We find

$$\eta = \ell nR = \tfrac{1}{2}\ell n(\rho^2 + z^2)$$

$$\xi = \theta = \tan^{-1}(\frac{\rho}{z}) \quad .$$

Because this is an instructive example, it is carried out in the Appendix. We evaluate for our Schwarzschild model problem all the terms which will come up in the two black hole problem.

Returning to the two black holes, we find that the Misner coordinates are achieved by placing two equal but opposite charges ($\ell n\zeta$ terms) at $z = \pm \coth \mu_0$. The field lines and equipotential surfaces of this simple charge distribution are Misner's coordinate lines for μ and θ. Cadez's coordinates are realized as the field lines and equipotential

surfaces for two equal charges with the same sign placed at $z = \pm \coth \mu_0$, together with enough higher multipole moments to make the throats lie on circles in the coordinates ρ and z.

The details are as follows.[7] Take the conformally flat line element:

$$ds^2 = a^2\psi^4(d\rho^2 + dz^2 + \rho^2 d\phi^2)$$

where $\Delta\psi = 0$. By image methods,[8] build up the matched initial conditions:

$$\psi = 1 + \sum_{n=1}^{\infty} \operatorname{csch}(n\mu_0)\left[\frac{1}{{}^{+}r_n} + \frac{1}{{}^{-}r_n}\right]$$

$${}^{\pm}r_n = [\rho^2 + (z - \coth n\mu_0)^2]^{\frac{1}{2}}$$

Now make a complex coordinate transformation:

$$\eta + i\xi = \chi(\zeta) \qquad \zeta = z + i\rho \qquad \zeta_0 = \coth(\mu_0)$$

Misner $\quad \chi(\zeta) = \ell n(\zeta+\zeta_0) - \ell n(\zeta-\zeta_0)$

Cadez $\quad \chi(\zeta) = \frac{1}{2}[\ell n(\zeta+\zeta_0) + \ell n(\zeta-\zeta_0)]$

$$+ \sum_{n=1}^{\infty} C_n \left(\frac{1}{(\zeta_0+\zeta)^n} + \frac{1}{(\zeta_0-\zeta)^n} \right)$$

The Jacobian of this map is

$$J = \left|\frac{d\chi}{d\zeta}\right|^2 = \left(\frac{\partial\eta}{\partial\rho}\right)^2 + \left(\frac{\partial\eta}{\partial z}\right)^2$$

since the Cauchy-Riemann conditions tell us that:

$$\frac{\partial\eta}{\partial z} = \frac{\partial\xi}{\partial\rho} \qquad \frac{\partial\eta}{\partial\rho} = -\frac{\partial\xi}{\partial z}$$

$$\frac{\partial^2\eta}{\partial z^2} = -\frac{\partial^2\eta}{\partial\rho^2} = \frac{\partial^2\xi}{\partial\rho\partial z} \qquad \frac{\partial^2\eta}{\partial z\partial\rho} = \frac{\partial^2\xi}{\partial\rho^2} = -\frac{\partial^2\xi}{\partial z^2} \quad .$$

This means that the new line element is:

$$ds^2 = a^2\psi^4\left(\frac{d\eta^2+d\xi^2}{J} + \rho^2 d\phi^2\right)$$

$$\rho = \rho(\eta,\xi) \qquad J = J(\eta,\xi) \qquad \psi = \psi(\eta,\xi)$$

By an easy calculation using $\coth^{-1}\eta = \frac{1}{2}\ell n\left(\frac{\eta+1}{\eta-1}\right)$, one finds

$$\eta = \mathrm{Re}\chi \quad \text{and} \quad \xi = \mathrm{Im}\chi$$

are just the bispherical coordinates used by Misner and Lindquist:

$$\coth\eta = (\rho^2 + z^2 + z_0^2)/(2zz_0) \qquad z_0 = \coth\mu_0$$
$$\cot\xi = (\rho^2 + z^2 - z_0^2)/(2\rho z_0)$$

and the Jacobian is

$$J = (\cosh \eta - \cos \xi)^2/z_0^2 \quad .$$

Since the inverse transformation shows

$$\rho = \sin \xi/(\cosh \eta - \cos \xi)$$

we can factor out J^{-1} to find the line element in Misner-Lindquist form:

$$ds^2 = a^2\tilde{\psi}^4(d\eta^2 + d\xi^2 + \sin^2 \xi \, d\phi^2)$$

$$\tilde{\psi}^4 = \psi^4 J^{-1} \quad .$$

The same procedure can be applied to the Cadez $\chi(\zeta)$ except that one must define $\rho = \rho(\eta,\xi)$ numerically. This process will be described in the next chapter. The important difference between the two coordinate systems is that the conformal map for Misner breaks down as:

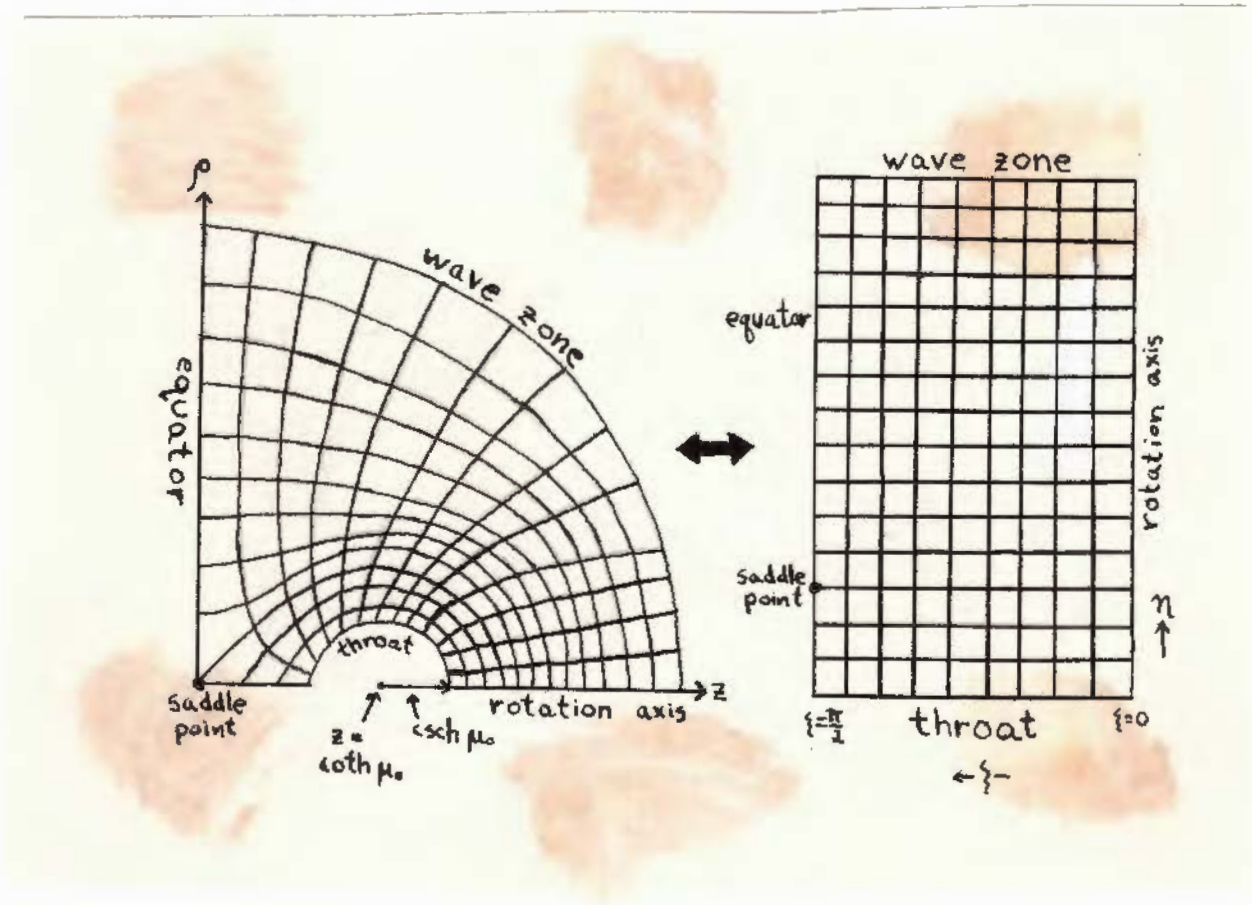
$$\left|\frac{d\chi}{d\zeta}\right| \to 0 \qquad \rho,z \to \infty$$

whereas for Cadez it breaks down as:

$$\left|\frac{d\chi}{d\zeta}\right| \to 0 \qquad \rho,z \to 0$$

This is why Misner's coordinates look so distorted as one gets farther away from the central region (squares are no longer mapped onto squares). For the Cadez system one only has the central saddle point to worry about (see figure 1). This distorts only the neighboring squares to the central point. Because the metric functions are regular at this point in the (z,ρ) system (as well as everywhere else), we can switch back and forth between the (η,ξ) and (z,ρ) coordinate systems to insure we are calculating in a "good" set of coordinates at all times.

Thus, we finally have a grid which schematically looks like this:



Notice that while the grid is curvilinear in (ρ,z), it is a simple rectangular mesh as viewed in (η,ξ) coordinates. Therefore, we will store quantities in the computer evaluated at the (η,ξ) points.

There are two free parameters (a,μ_0) which appear in the line element. One has the dimensions of length (a) and one is dimensionless (μ_0). We have pulled a^2 out of the line element so that the coordinates (ρ,z) are dimensionless like the coordinates (η,ξ). Let us relate the parameters (a,μ_0) to the physical parameters (M,L) where M is the total mass of the system as measured at infinity and L is the proper distance along the spacelike geodesic connecting the two throats, i.e. twice the proper distance from the saddle point to the nearest edge of the throat. As shown in Misner and in Lindquist

$$M = 4a \sum_{n=1}^{\infty} \operatorname{csch}(n\mu_0)$$

$$L = 2a[1 + 2\mu_0 \sum_{n=1}^{\infty} n \operatorname{csch}(n\mu_0)] \quad .$$

We can see the result for M in two ways. First, since we have $\psi \to 1 + M/2r$ as $r \to \infty$, we can just note that as $\rho,z \to \infty$, ${}^{\pm}r_n \to r \to \infty$ and so:

$$\psi \to 1 + \frac{1}{r} \sum_{n=1}^{\infty} \operatorname{csch}(n\mu_0) \quad .$$

Second, we can apply the method of O'Murchadha and York which rewrites the mass formula of ADM in the form:

$$M = -\frac{1}{2\pi}\oint \nabla\Psi \cdot ds$$

where the integral is taken over a 2-surface at spatial infinity. Since we have $\Psi = \Psi(\rho,z)$ we can write the integrand as:

$$\nabla\Psi \cdot ds = (\partial_r \Psi) r^2 \sin\theta d\theta d\phi$$

$$= [(\partial_\rho \Psi)\sin\theta + (\partial_z \Psi)\cos\theta] r^2 \sin\theta d\theta d\phi \ .$$

Explicitly taking the derivatives and then letting $\pm r_n \to r \to \infty$, we integrate over the angles and again obtain the formula for M.

As shown in Misner (1960), Wheeler (1961), Lindquist (1963), and Brill and Lindquist (1963), the two throats distort themselves in a manner suggesting a Newtonian tidal deformation. This is to be expected since, as we have shown, the only nonzero component of the curvature tensor for a vacuum time-symmetric slice is E_{ab}. One can define an interaction energy for this bound system as

$$m_{int} = M - 2m = -4a \sum_{n=1}^{\infty} (n-1)\,\mathrm{csch}(n\mu_0)$$

$$m = 2a \sum_{n=1}^{\infty} n\,\mathrm{csch}(n\mu_0)$$

where m is the "bare" mass assigned to each throat (Lindquist, 1963). Note that this interaction energy grows more and more negative as $\mu_0 \to 0$.

From the work of Hawking and others we now know that the throats are not the surfaces of the black holes. The event horizon will lie outside the throats. In the initial time-symmetric surface, the throats are minimal surfaces and, thus, marginally trapped surfaces. As mentioned earlier, when the holes are close enough together, a new minimal surface appears surrounding the entire system.[9] This tells us, just by looking at the initial surface, that the event horizon for the two black holes will merge and form a distorted black hole as they evolve.

It is important for us to know when such a merging has occurred. Cadez[10] has found the value of μ_0 at which the new minimal surface first appears ($\mu_0 = 1.362$) and has calculated its position and area. We will see in a moment that this enables us to set upper bounds on the amount of radiation which might be produced by the evolution. In table 1, I give values of M, L, L/M, m/M, m_{int}/M as functions of μ_0. For reference, our computer calculations have been carried out from initial data with $\mu_0 = 2$.

We turn now to the evolution of the system. For an overall view of the expected development of the horizon structure see Hawking (1973), especially his picture on page 35. As one can see, there are essentially three phases in

Table 1

μ_0	M	L	L/M	m/M	M_{int}/M
1.0	5.14	9.87	1.92	0.77	-0.53
1.5	2.39	6.58	2.75	0.64	-0.28
2.0	1.27	4.94	3.88	0.58	-0.15
2.5	0.72	3.96	5.50	0.54	-0.09
3.0	0.42	3.33	7.92	0.53	-0.05

this evolution. The first is just an in-flight period while the holes are far apart but moving towards each other. The second is when they get close enough to each other that they very rapidly tidally distort one another. The final phase is when they are coalescing. During this last phase one has a single black hole with huge nonspherical deformations.

During each of these phases we expect gravitational radiation to be produced. However the mechanism is somewhat different in each case. For the in-flight radiation, we can think of the system as two point masses falling towards each other. This gives a time-changing quadrupole moment. As they near each other, their finite dimensions come into play, and we can imagine each hole to have a "quadrupole" moment which changes very rapidly as the holes distort one another. Finally, with the coalescence, the whole chaotic strong field region is churning and vibrating. One may very well expect a strong burst of radiation as the equatorial sections of the event horizon expand outward to form a spherical black hole. After this big pulse, a long train of "ringing" waves may occur,[11] gradually dying out until only a spherical black hole is left surrounded by a region of radiation heading out to infinity.

We can use various techniques to estimate the importance of these phases for producing radiation. Let us assume that the holes are projected inward from infinity with velocity v. Then by energy conservation:

$$E_1 + E_2 = E_f + E_{rad} \qquad (c = 1)$$

$$\sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2} = M_f + E_{rad}$$

where E_1 and E_2 are the energies of the two holes, E_f is the final energy of the spherical black hole, and E_{rad} is the energy carried off by radiation. We can combine this with Hawking's theorem that the total surface area of black holes must increase:

$$A_1 + A_2 \leq A_f$$

where $A_i = 16\pi M_i^2$, to obtain an upper limit on the efficiency of radiation. For two equal masses m projected inward with

$$\gamma = (1 - v^2)^{-\frac{1}{2}}$$

we find:[12]

$$\eta_0 = \frac{E_{rad}}{2m\gamma} \leq 1 - \frac{1}{\gamma\sqrt{2}} \begin{cases} \nearrow 1.00 & \gamma \to \infty \quad v \to 1 \\ \searrow 0.295 & \gamma \to 1 \quad v \to 0 \end{cases} .$$

The reason the efficiency rises so fast as $v \to 1$ is that the colliding black holes become "Lorentz contracted" and appear to be more and more like colliding plane waves.[13] Their energy is almost totally bound up in kinetic energy which presumably will be radiated leaving only the "rest mass" in

the final black hole.

For two black holes which are initially bound, upper limits can also be calculated, but they are not as stringent since they depend on using the apparent horizons instead of the event horizons. The notion is that the bound system can be approximated by the Misner initial conditions. In such a spacelike slice we can measure the area of the minimal surfaces and hope this approximates the area of the horizon. We then use area theorems as before to estimate the radiation. Gibbons and Schutz carried out this program for the minimal surfaces at the throats. They found the efficiency must decrease as the holes are moved together. Unfortunately they made an error in calculating L so their graph must have its scale for (M/L) corrected. (There was no error in their calculation of the efficiency as a function of μ_0.) The numbers are graphed in Figure 2.

Čadež has since found the location of the third minimal surface surrounding the two throats for $\mu_0 \leq 1.362$. It enables us to place on the efficiency curve a "cutoff" which is much more stringent. The probable upper limit is sketched in. As one can see, the upper limit is about 5% for $\mu_0 = 2$. For this reason we should not expect to see very much "wiggling" in the resulting curvatures for our computer runs.

Going back to the parabolic infall where Hawking gives an upper limit of 29.5%, we can estimate the in-flight radiation as follows. We will approximate the system by two

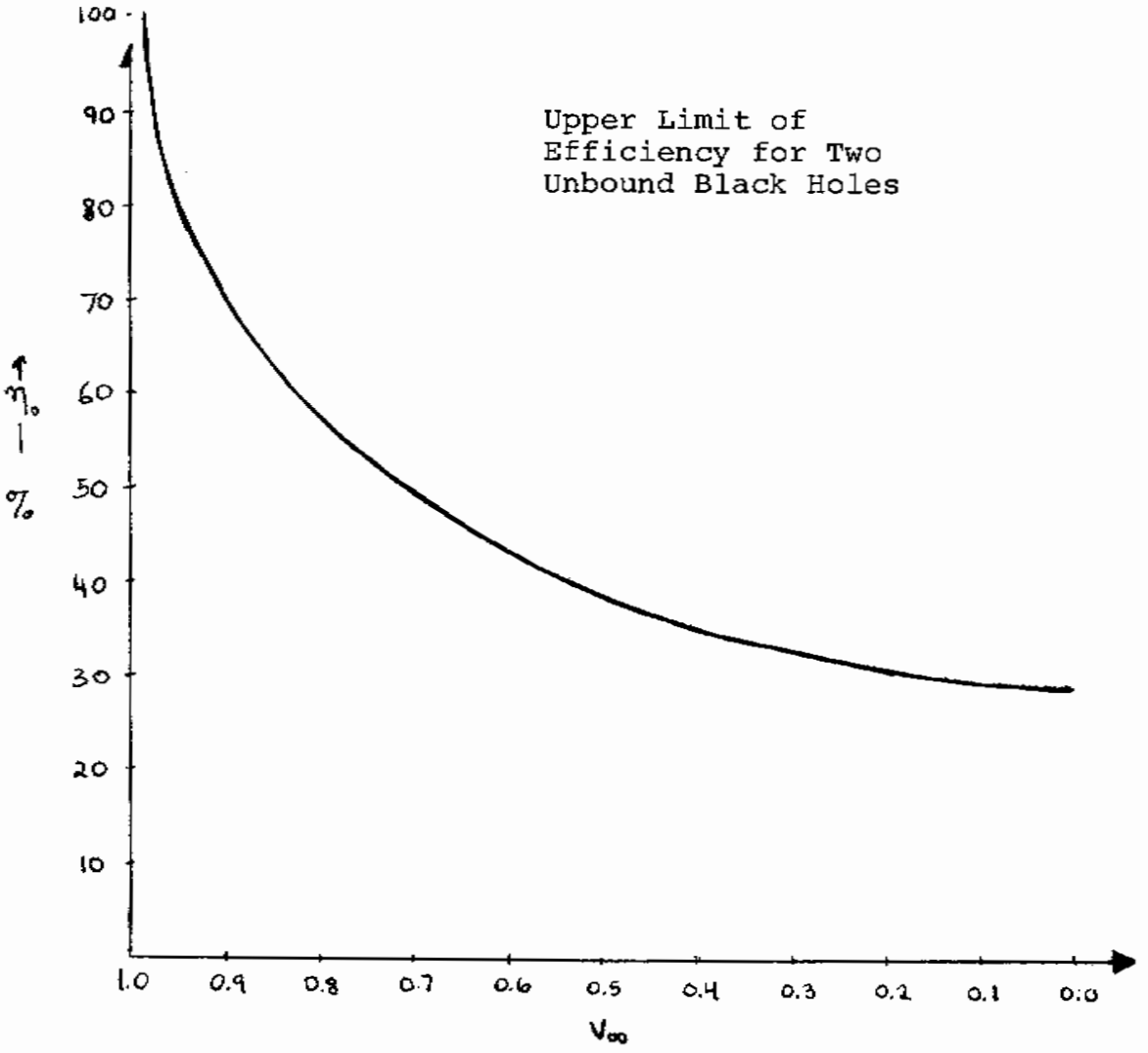

Upper Limit of
Efficiency for Two
Unbound Black Holes
100
90
80
70
60
50
40
30
20
10
% — $\frac{E_{out}}{E_{in}}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
V_∞

Figure 2.a

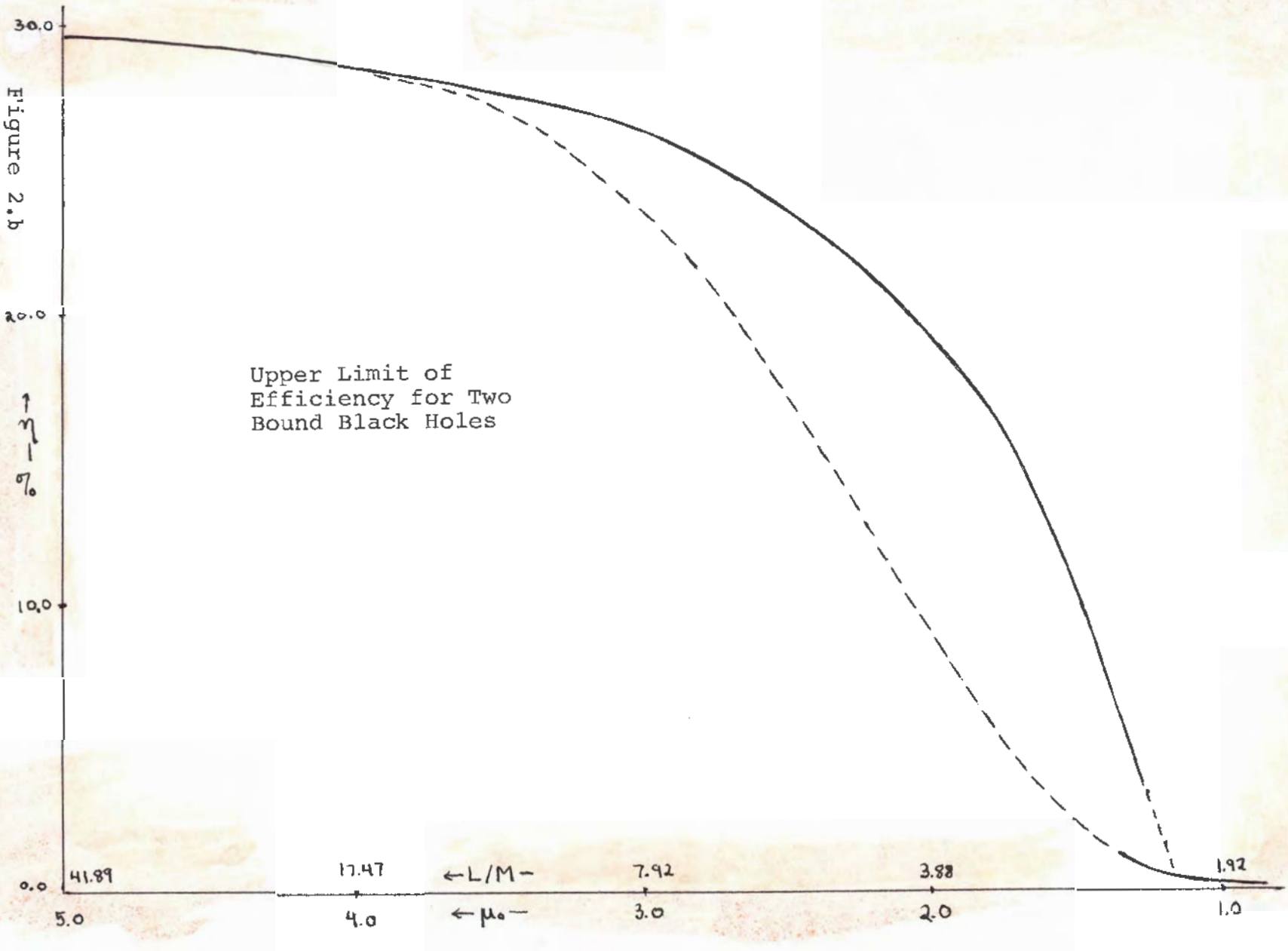

30.0
20.0
10.0
0.0
η — %
Upper Limit of
Efficiency for Two
Bound Black Holes
41.89
17.47
7.92
3.88
1.92
←L/M—
5.0
4.0
3.0
2.0
1.0
←μ₀—

Figure 2.b

point particles of masses m_1 and m_2. In the center of mass system, we have the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ and the total mass $M = m_1 + m_2$. Using Newton's law of motion with the force given by Newton's law of gravity one finds that the relative separation varies with time ($t \to 0$ as $r \to 0$)

$$r = \left(\frac{9M}{2}\right)^{\frac{1}{3}} t^{\frac{2}{3}} \qquad \mu\ddot{r} = \frac{M\mu}{r^2} \qquad G = c = 1 \quad .$$

For axially symmetric systems, one can find the power radiated by the time-changing quadrupole to be:[14]

$$I(t) = \frac{2}{15}Q^2 \qquad Q = \mu\frac{\partial^3}{\partial t^3}(r^2) \quad .$$

To get the total energy we integrate over the time of infall. This will diverge, so we cut off the integral when $\dot{r} \to 1$ and $r \to 2M$, that is, just as the two holes touch and our approximation breaks down anyway. Also, we may fold in a red shift "fudge factor" to degrade the radiation produced in the strong field region. The result is:

$$E_{rad} = \int_{-\infty}^{t} I(t)\left(1 - \frac{\alpha M}{r}\right)dt \quad \begin{matrix} \xrightarrow{\alpha=0} & 0.019\ (\mu^2/M)c^2 \\ \xrightarrow{\alpha=1} & 0.011\ (\mu^2/M)c^2 \\ \xrightarrow{\alpha=2} & 0.004\ (\mu^2/M)c^2 \end{matrix}$$

We can calibrate this result by comparing it with a fully relativistic calculation[15] in the limiting case

where $\mu << M$ and we have a small mass μ falling into a large black hole of mass M. For this situation one finds:

$$E_{rad} = 0.011(\mu^2/M)c^2 \qquad (\mu << M)$$

which is right on target. Throwing caution and rigor to the winds, we investigate the case where $m_1 = m_2 = m$. Now we have:

$$\mu = \tfrac{1}{2}m \qquad M = 2m \qquad \mu^2/M = \frac{1}{16} M$$

$$(m_1 = m_2)$$

$$E_{rad} = 0.01(\frac{1}{16} Mc^2) = 0.0005Mc^2$$

This is to be compared with Hawking's upper limit of

$$E_{rad} = 0.2950\ Mc^2 \quad \text{(Hawking)} \quad .$$

Thus, Hawking's upper limit is 600 times as high as our estimate[16] for the in-flight radiation. This is one of the main reasons we feel that if any appreciable amount of radiation is produced by the collision, it will come from the last two phases.

Estimates for these phases are more difficult. Mashhoon (1972) has calculated tidal gravitational radiation for neutron stars falling into black holes. If one uses a neutron star of black hole "density" and extent in Mashhoon's formulas, one comes out with maybe 1.-5.% as an upper limit.

This leaves the great bulk of Hawking's upper limit up to coalescence radiation. Press and Teukolsky[17] have shown that a very large "perturbation" of a Schwarzschild black hole could cause it to radiate as much as 10% of its rest mass. They stress that this calculation violates the perturbation assumptions of their equations. Nonetheless, it gives one some small hope that a highly deformed black hole may be able to radiate a fair fraction of its mass-energy away.[18]

The issue is of central importance if black hole collisions are to be of astrophysical importance. Of course, one does not expect isolated black holes to collide frequently enough to be observed. However, in the nuclei of galaxies[19] dense aggregations of stars and dust may develop.[20] If gravitational collapse occurs sufficiently often, then a relativistic cluster[21] of collapsed objects could form. Such a cluster would be a powerful generator of pulses of gravitational radiation.[22] Further, it would be recurring since the black hole which is formed from the collision of two others can collide again. It would not emit any electromagnetic radiation, and the frequency and energy of the pulse would be about that detectable by a Weber-type antenna. To be sure, it is still not decided today whether Weber is in fact receiving gravitational pulses, but there are some recent astronomical measurements which tend to indicate a mass loss of 50-100 M_0/year for the galaxy.[23] This whole scenerio should be investigated further.

Other computations which could be performed using our general approach are outlined now. The two black hole collision could be investigated for its sensitivity to initial conditions, as mentioned above. By dropping the equatorial reflection plane symmetry and the symmetry through the throat, one could evolve unequal masses. If our "generalized superposition" approach to the initial value problem works we can make the holes collide with greater and greater velocity to see 1) if large amounts of radiation can be produced and studied or 2) if the horizon is stable, i.e. if one can form a naked singularity. Finally, the collision of two rotating black holes might be attempted although the initial value problem is unsolved as yet. Because nonaxisymmetric problems are too complicated for computers to probe in detail, we can not calculate the "off-center" collision. However, we may be able to guess what the qualitative changes are from study of simpler models.

Moving away from the two body problem, there are many different collapse problems one can imagine. The simplest is a nonspherical collapse to a pancake of a star which is initially nonspherically deformed. Recent work of Thuan and Ostriker (1974) for the Newtonian collapse of a star shows that a fair amount of radiation may be produced. The next step would be to perform a fully relativistic calculation of this process for both the nonrotating and rotating cases. Of considerable interest is whether horizons form in

these highly nonspherical problems (Thorne, 1972). In the rotating case, the calculation would do a great deal to decide whether the Kerr geometry is the final state of realistic collapse.

Another area of interest lies in the collapse of bumpy stars and the possible production of gravitational run-away stars. In the first case, one imagines a star with random axisymmetric bumps or multiple moments which then radiate away as the star collapses. Since such bumps normally grow in time as the collapse proceeds, it will be very interesting to see how radiation smooths out or further accentuates these bumps. Tom Criss at Texas is currently working on the initial value problem for such a star. If the bumps are such that the star has a time changing octupole moment, Bekenstein (1973) has shown that the gravitational radiation will carry away momentum and the neutron star or black hole that is left will become a "run-away." In particular, he finds speeds sufficiently high to be observable astronomically. The details of this process could be analyzed by the numerical formalism we have outlined.

References

Chapter VII

1. For a discussion and references see Synge (1960) or Cooperstock (1974).
2. This was known to Lichnerowicz (1944) and is physically self-evident since only one sign of mass occurs in nature. For modern proofs see Gibbons (1974) and Müller Zum Hagen and Siefert (1973). For the balanced charged black holes see Gautreau, Hoffman, and Armanti (1972).
3. See Wheeler (1961) for a review and set of references on EIH. This article also reviews the geometrodynamical approach to the problem of motion.
4. See a review of it in Bruhat (1962).
5. Smarr, Ostriker, Criss--work in progress.
6. I wish to thank Adam Drobot, James Wiley, Tom Criss, Steve Christensen, Mrs. Florence Turck, and Al Jackson for their advice on many programming problems.
7. For the original derivation of these coordinates see Čadež (1971).
8. Misner (1960, 1963), Lindquist (1963).
9. Brill and Lindquist (1963), Čadež (1974), Hawking (1973).
10. Čadež (1974)--private communication.
11. Press (1971).
12. For $\gamma = 1$ this calculation was first performed by

Hawking (1971). For $\gamma > 1$ it was done by Bekenstein and Smarr (unpublished).

13. Pirani (1959), Matzner and Nutku (1974), Penrose and Kähn (1971), Szekeres (1972).
14. See for instance, Bekenstein (1973).
15. Davis, et al. (1971).
16. For similar calculations see Zeldovich and Novikov (1971), Zerilli (1970), and Kafka (1970).
17. 1973--private communication.
18. See also Gibbons (1972).
19. See Burbidge (1970) for a review.
20. Spitzer and Saslaw (1966), Sanders (1970).
21. Zeldovich and Podurets (1966), Fackerell, Ipser, and Thorne (1969), Zeldovich and Novikov (1971).
22. Kafka (1969, 1970, 1972), Greenstein (1969), Bertolli and Cavaliere (1971), Hawking (1972).
23. Clube (1973) p. 460.

Appendix

Chapter VII

We transform the Schwarzschild initial data set into conformal coordinates like those we use in the two black hole problem. In particular, this should give the (η,ξ) dependence of the two black hole initial data as $\eta \to \infty$. First, we write Schwarzschild in isotropic coordinates and then convert these coordinates to dimensionless ones by letting:

$$\rho \to \frac{M}{2}\rho \qquad z \to \frac{M}{2}z \qquad R \to \frac{M}{2}R$$

where (z,ρ) are cylindrical coordinates and R is the isotropic radial coordinate. This gives the line element:

$$ds^2 = a^2\psi^4(d\rho^2 + dz^2 + \rho^2 d\phi^2) \qquad a = \frac{M}{2} \qquad \psi = 1 + R^{-1} \ .$$

We now apply a complex conformal coordinate transformation:

$$\eta + i\xi = \chi(\zeta) = \ell n\zeta \quad \text{where } \zeta = z + i\rho \ .$$

This gives us new radial (η) and angular coordinates (ξ):

$$\eta = \ell nR = \tfrac{1}{2}\ell n(\rho^2 + z^2) \qquad \xi = \theta = \tan^{-1}\frac{\rho}{z}$$

We can directly verify the Cauchy-Riemann equations:

$$\partial_\rho \eta = -\partial_z \xi = \rho/(\rho^2 + z^2) \qquad \partial_z \eta = \partial_\rho \xi = z/(\rho^2 + z^2)$$

$$J = (\partial_\rho \eta)^2 + (\partial_z \eta)^2 = (\rho^2 + z^2)^{-1} = R^{-2} = e^{-2\eta}$$

In our new coordinates, the line element becomes:

$$ds^2 = a^2\psi^4(R^2 + R^2 d\Omega^2) \qquad \psi = 1 + R^{-1}$$

$$= a^2\psi^4\left(\frac{d\eta^2 + d\xi^2}{J} + \rho^2 d\phi^2\right) \qquad \psi = 2e^{-\eta/2}\cosh\frac{\eta}{2}$$

$$= a^2\tilde{\psi}^4(d\eta^2 + d\xi^2 + \sin^2\xi d\phi^2) \qquad \tilde{\psi} = \psi J^{-1} = 2\cosh\frac{\eta}{2} \ .$$

The Christoffel symbols for the conformal metric (not conformally flat if we use $\tilde{\psi}$ instead of ψ!) are (we set $a = 1$ from here on):

$$G_{\phi\phi}{}^{\xi} = -\sin\xi\cos\xi \qquad G_{\phi\xi}{}^{\phi} = \cot\xi \qquad \text{others vanish} \ .$$

The only derivatives of $\tilde{\psi}$ that are nonzero are:

$$\tilde{\psi}_{,\eta} = \sinh\frac{\eta}{2} \qquad \tilde{\psi}_{,\eta\eta} = \tfrac{1}{2}\cosh\frac{\eta}{2} \ .$$

Combining these, using the formulas for conformally related metrics from Chapter IV, yields the Christoffel symbols of the

full metric:

$$\Gamma_{\eta\eta}{}^{\eta} = \Gamma_{\eta\xi}{}^{\xi} = -\Gamma_{\xi\xi}{}^{\eta} = \tanh\frac{\eta}{2} \qquad \Gamma_{\phi\phi}{}^{\eta} = -\tanh\left(\frac{\eta}{2}\right)\sin^2(\xi)$$

$$\Gamma_{\phi\phi}{}^{\xi} = -\sin\xi\cos\xi \qquad \Gamma_{\phi\xi}{}^{\phi} = \cot\xi \qquad \text{others vanish} \ .$$

The Riemann tensor, Ricci tensor, and Riemann scalars of the conformal geometry are:

$$R_{\phi\xi\phi\xi} = \sin^2\xi \qquad \text{others vanish}$$

$$R_{\xi\xi} = 2 \qquad R_{\phi\phi} = \sin^2\xi \qquad \text{others vanish}$$

$$R = 2 \ .$$

For the full geometry we find:

$$R^{\phi\xi}{}_{\phi\xi} = \tilde{\psi}^{-4} - 4\tilde{\psi}^{-6}\tilde{\psi}_{\eta}{}^{2}$$

$$R^{\eta\xi}{}_{\eta\xi} = R^{\eta\phi}{}_{\eta\phi} = -2\tilde{\psi}^{-6}$$

$$R_{\eta\eta} = -\mathrm{sech}^2\frac{\eta}{2} = -2R_{\xi\xi} = -2\sin^{-2}\xi\, R_{\phi\phi}$$

$$R = 0$$

Using the 3-Riemann tensor components we can calculate the

curvature invariant I for the 4-Riemann tensor:

$$I = {}^4R_{abcd}\,{}^4R^{abcd} = 192\,\tilde{\psi}^{-12}$$

The lapse function which is zero at the throat ($\eta = 0$) and goes to one at infinity is just

$$\alpha = (1 - R^{-1})/(1 + R^{-1}) = \tanh\frac{\eta}{2}$$

Its covariant derivatives are:

$$\alpha_{\cdot\eta\eta} = -\operatorname{sech}^2\frac{\eta}{2}\tanh\frac{\eta}{2} = -2\alpha_{\cdot\xi\xi} = -2\sin^{-2}\xi\,\alpha_{\cdot\phi\phi}$$

By substitution one can check that this satisfies the lapse equation:

$$\Delta\alpha = 0$$

Furthermore, the only nonzero terms in the evolution equations cancel each other so that $\pounds_{\alpha n}K_{ab} = \pounds_{\alpha n}h_{ab} = 0$:

$$\alpha_{\cdot ij} - \alpha R_{ij} = 0$$

Finally, one can check that the "radiation components" of Q_{ab} are given by:

$${}^4R_{t\theta t\theta} = \tfrac{1}{2}\tanh^2\frac{\eta}{2}\operatorname{sech}^2\frac{\eta}{2} \qquad {}^4R_{t\theta t\phi} = 0$$

which qualitatively have the shape we observe in the two black hole initial slice if we move from the throat out the z-axis.

Chapter VIII

We will describe in this last chapter how to build a computer code which will calculate an axisymmetric time-dependent solution of Einstein's gravitational field equations. We will concentrate on the nonrotating, vacuum case with zero shift vector and, in particular, on the two black hole collision. However, the generalization to these other cases should be straightforward even if messy. I will not give a listing of the codes I developed since many of their features are peculiar to the UT-CDC 6600 computer system (copies are available on request). Rather, I shall attempt to describe the approach we took and a number of the numerical difficulties we ran into.

An axisymmetric system is one for which the field variables depend only on two space variables and time. The cyclic coordinate will be assumed to be associated with a Killing vector possessing closed orbits, i.e. an axial Killing vector $\partial/\partial\phi$. The three metric for each space slice has six unknown functions. For the nonrotating case we can drop two of them and consider the other four to be functions of two space coordinates (η,ξ) or (z,ρ) spanning each ϕ = constant slice. Since the initial metric is conformally flat with the conformal factor ψ known, we will always factor ψ^4 out of the metric and evaluate its derivatives analytically. This means we will evolve only the conformal metric in keeping

with York's emphasis. Because of the coordinate transformation described in the last chapter, we know $\rho^2 = \rho^2(\eta,\xi)$ so we will always factor it out of the $d\phi^2$ term. This leaves us with the metric tensor:

$$\gamma_{ij} = \psi^4 \begin{pmatrix} A & C & 0 \\ C & B & 0 \\ 0 & 0 & \rho^2 D \end{pmatrix} \begin{matrix} \eta & z \\ \xi & \rho \\ \phi & \phi \end{matrix}$$

$$\begin{matrix} \eta & \xi & \phi \\ z & \rho & \phi \end{matrix}$$

As we shall see shortly, we can think of A, B, C, D as functions either of (η,ξ) or of (z,ρ) because of the general covariance of the Einstein equations.

If such a nonrotating axisymmetric system generates gravitational radiation, it will be of only one polarization. We can see this by examining $Q_{\theta\theta}$ and $Q_{\theta\phi}$ which become the real and imaginary parts of the Weyl tensor component Ψ_0 as one goes far away from the source. We derived their relation to K_{ab} and R_{ab} at the beginning of this thesis:

$$Q_{ab} = K_{ca}K_b{}^c - KK_{ab} - R_{ab} = \alpha^{-2}R_{tabt} \quad .$$

If $a = \theta$ and $b = \phi$, then $R_{\theta\phi} = K_{\theta\phi} = 0$ and $K_{c\theta}K_{d\phi}h^{cd} = 0$ because $h^{\theta\phi} = 0$. This means that the "x" polarization state vanishes, leaving only the "+" state. Later, we will show how

${}^4R_{t\theta\theta t} = \alpha^2 Q_{\theta\theta}$ varies in time for the two black hole collision.

We remark on an interesting property of the algebraic type of the Weyl tensor for axisymmetric solutions. Consider the four principal null directions for the Weyl tensor. Now if two or more of these directions line up at a given event, then the Weyl tensor is said to be algebraically special. On a spacelike hypersurface each null vector can be resolved into a part along the normal vector to the surface and a part lying in the surface. If one is on the axis of symmetry then the only available spatial direction lies along the axis. If the spatial part was tilted with respect to the axis, it would pick out a preferred ϕ value, which is not allowed. Thus, the principal null direction must lie along the axis and this means only two directions are possible. Therefore, on the 2-plane generated by the symmetry axis moving in time, the Weyl tensor is algebraically special of type {2,2}, {3,1}, {4}, or {-}.[1] At an event like the saddle point where one has the intersection of a symmetry axis and an equatorial reflection plane, the type must be {2,2} or {-}. I calculated the type for the symmetry axis of the balanced charged black holes[2] and found the type to be {2,2} everywhere on the axis.

As one last bit of lore about the saddle point, I note that the world line generated by the saddle point must be geodesic. This is because at the saddle point all first

derivatives vanish, including those of α. From our formula for a_b, the proper acceleration of this world line, we see that if $\alpha_{,i} = 0$, then $a_b = 0$. However, this does not mean the saddle must free-fall into the singularity. As noted before, when $\alpha \neq 1$, the time coordinate t labeling each slice is not proper time along the normal curves. For a geodesic, this means t is not an affine parameter of the curve, and therefore the time t may move more slowly than the free fall proper time τ. In other words, if we let $\alpha \to 0$ in the saddle region, then we can slow down the rate of advance of the hypersurfaces there to any desired degree.

Now let us consider how to set up our mesh[3] and how we will difference spatial derivatives. We outlined the complex conformal mapping technique in the last chapter. To implement it in our FORTRAN program, we use two subprograms. The first program is a least squares routine which makes the real part of our complex χ equal to a constant on the throat. In (z,ρ) coordinates, the throat is the circle:

$$\rho^2 + (z \pm \coth \mu_0)^2 = \operatorname{csch}^2 \mu_0 \ .$$

Since the equator ($z = 0$) is a mirror plane and the z-axis is a rotation axis, all our computations can be performed in the quadrant $z \geq 0$, $\rho \geq 0$. Therefore, we only need to require that:

$$\mathrm{Re}(\chi) = \eta = \eta_0$$

on the semicircle in this quadrant. (η_0 = constant is not related to the efficiency η_0 we discussed in the last chapter.)

The fit is made by performing a least squares fit on the NMAX real coefficients c_n (n = 1, NMAX) which appear in the power series for $\chi(\zeta)$. Since we only need to find real c_n, we only have to specify $\mathcal{R}e(\chi)$ on the circle. It was found that our fit was much tighter if we also let η_0 be determined by the least squares method. The resulting curve of $\eta(z,\rho)$ vs. η_0 where z and ρ are constrained to lie on the throat is shown in Figure 3. The chisquare is defined here to be

$$\text{chisquare} = \int_{z_-}^{z_+} (\eta-\eta_0)^2 dz$$

and has the value 3.6×10^{-14}. The maximum $|\eta - \eta_0| = 10^{-6}$ where η varies from $\eta_0 = -0.27$ on the throat to typically $\eta = 2.30$ at the wave zone. The graph shown, as all other results we display herein, has been calculated assuming $\mu_0 = 2.0$.

With the c_n chosen (for our runs NMAX = 5) the next program takes over. We first choose the integral number of "equipotential surfaces" we want between the throat and the saddle point. This sets the mesh spacing in (η,ξ). That is if NUM = the number of $\Delta\eta$ steps between the throat ($\rho = 0$, $z_t = \coth \mu_0 - \operatorname{csch} \mu_0$) and the saddle ($z = \rho = 0$), then the

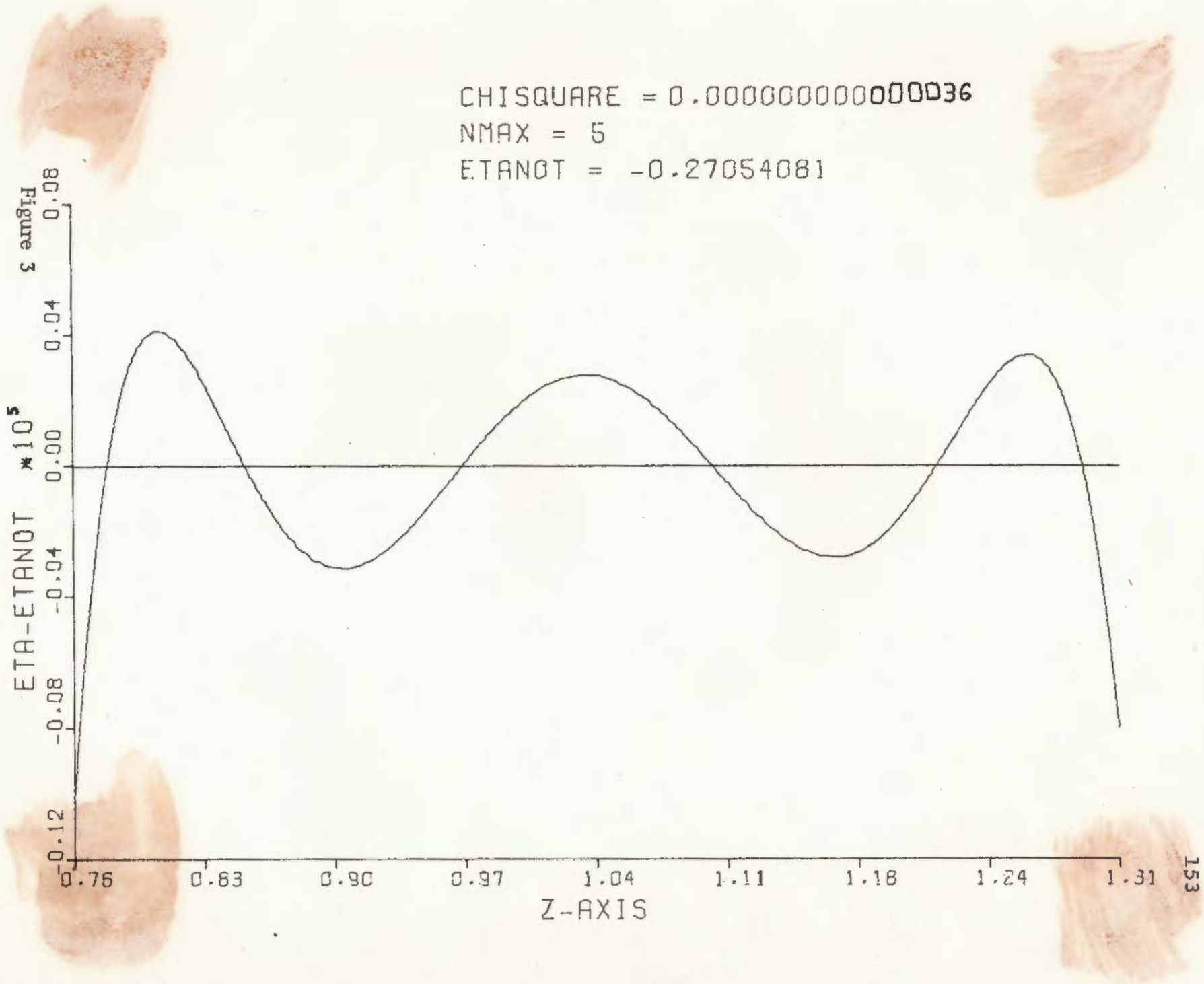

CHISQUARE = 0.000000000000036
NMAX = 5
ETANOT = -0.27054081
ETA-ETANOT *10^5
0.08
0.04
0.00
-0.04
-0.08
-0.12
0.76
0.83
0.90
0.97
1.04
1.11
1.18
1.24
1.31
Z-AXIS

Figure 3

size of $\Delta\eta$ is given by

$$\Delta\eta = (\eta(0,0) - \eta(z_t,0))/NUM \quad .$$

We then select the number of $\Delta\xi$ steps from the z-axis ($\xi = 0$) to the equator ($\xi = \pi/2$), by making $\Delta\xi$ as close to $\Delta\eta$ as we can while maintaining an integral number of $\Delta\xi$ steps from 0 to $\pi/2$. With $\Delta\eta$ and $\Delta\xi$ selected, we then determine the (z,ρ) values for each (η,ξ) grid point. We start at the throat ($\eta = \eta_0$) and march outward INF*$\Delta\eta$ steps. For each "radial" value of η we swing around INT*$\Delta\xi$ steps from $\xi = 0$ to $\xi = \pi/2$. To determine (z,ρ) for a given $(\bar{\eta},\bar{\xi})$ we call on the IMSLIB routine ZANLYT which solves for the zero of the complex function:

$$\chi(z + i\rho) - (\bar{\eta} + i\bar{\xi}) \quad .$$

Since we need a first approximation to (z,ρ) for the routine to converge quickly, we always use the last grid point calculated as an initial guess for the next grid point.

Now that we have $z(\eta,\xi)$ and $\rho(\eta,\xi)$ stored in arrays of size INF*INT, we can evaluate the conformal factor ψ and its spatial derivatives at each point of the (η,ξ) grid. The power series for $\psi(z,\rho)$ is differentiated analytically with respect to (z,ρ) to form ψ_ρ, ψ_z, $\psi_{\rho\rho}$, $\psi_{z\rho}$. Similarly α and its derivatives are evaluated analytically on the initial

surface and stored. Finally, the Jacobian coefficients η_ρ, η_z, η_{zz}, $\eta_{z\rho}$ are calculated by taking the real and imaginary parts of the power series for $d\chi/d\zeta$ and $d^2\chi/d\zeta^2$. We always use the Cauchy-Riemann conditions to change ξ_z to $-\eta_\rho$, etc. All the calculated quantities are then sent out onto magnetic tape for storage.

The initial choice for α is at our disposal, of course. One could try $\alpha = 1$ everywhere or some algebraic condition on α with $\alpha \to 1$ at infinity. Čadež ran such evolutions and reports some results in his thesis (Čadež-1971). With α's which are not zero on the throat one has to have a scheme to obtain α at one step inside the throat $(\eta_0 - \Delta\eta)$. Čadež shows how to use the mirror isometry through the throat to handle this problem. We decided to stick to our maximal approach so we chose α to be a solution of $\Delta\alpha = 0$ on the initial surface. (The time symmetry implies $R = 0$.) Čadež has solved analytically for the α which obeys the boundary conditions $\alpha \to 0$ on the throat:

$$\alpha\psi = 1 + \sum_{n=1}^{\infty} (-1)^n \operatorname{csch}(n\mu_0)\left(\frac{1}{+r_n} + \frac{1}{-r_n}\right) .$$

This choice of α corresponds to the α on the Schwarzschild t = constant slices:

$$\alpha\psi = 1 - \frac{M}{2r} .$$

Now this illustrates one of the many differences between one black hole slices and two black hole slices. Notice that our maximal α for the two black holes obeys the same boundary conditions as the maximal α for the one black hole, but $\alpha\psi \to 1 - M/2r$ as $r \to \infty$. The α which does behave as the Schwarzschild α in its asymptotic properties is

$$\alpha\psi = 1 - \sum_{n=1}^{\infty} \operatorname{csch}(n\mu_0)\left(\frac{1}{+r_n} + \frac{1}{-r_n}\right)$$

but $\alpha < 0$ on the throat. This makes it clear that there is not always an unambiguous choice for carrying quantities from the one black hole to the two black hole case. In particular, the fact that α's asymptotic behavior does <u>not</u> convey information about the mass of the system, even on maximal slices, is simply an example of Misner's (1963) argument on "mass and goo."

At this stage we have our mesh and our initial data set up. The metric functions are $A = B = D = 1$, $C = 0$, where $A = f_{zz}$, $B = f_{\rho\rho}$, $C = f_{z\rho}$, and $D = f_{\phi\phi}$ and f_{ij} is the conformal metric:

$$f_{ij} = \gamma_{ij}/\psi^4 \qquad f_{\phi\phi} = \gamma_{ij}/\psi^4\rho^2 \quad .$$

As soon as evolution begins A,B,C,D will begin to vary over the mesh. We now have to decide how to find their spatial derivatives by differencing because we no longer have analytic

expressions for them. Throughout our programs we use the standard second-order accuracy center differences:

$$\partial_\eta H(I,J) = \frac{H(I+1,J)-H(I-1,J)}{2\Delta\eta} \qquad \partial_\xi H(I,J) = \frac{H(I,J+1)-H(I,J-1)}{2\Delta\xi}$$

$$\partial_{\eta\eta} H(I,J) = \frac{H(I+1,J)-2H(I,J)+H(I-1,J)}{(\Delta\eta)^2}$$

$$\partial_{\xi\xi} H(I,J) = \frac{H(I,J+1)-2H(I,J)+H(I,J-1)}{(\Delta\xi)^2}$$

$$\partial_{\eta\xi} H(I,J) = \frac{H(I+1,J+1)-H(I+1,J-1)-H(I-1,J+1)+H(I-1,J-1)}{4\Delta\eta\Delta\xi}$$

where H(I,J) is some function on the (η,ξ) grid. Now the computer thinks the (η,ξ) grid is rectangular (see illustration in the last chapter), so it always takes derivatives in the (η,ξ) grid. However, we want to evolve the metric functions γ_{zz}, $\gamma_{z\rho}$, etc. in the (z,ρ) coordinates (they are regular everywhere). Therefore, we always take (z,ρ) derivatives by the chain rule:

$$\partial_\rho = \eta_\rho \partial_\eta + \xi_\rho \partial_\xi = \eta_\rho \partial_\eta + \eta_z \partial_\xi \quad ,$$

etc., which is why we store $(\eta_\rho, \eta_z, \eta_{zz}, \eta_{z\rho})$ on the (η,ξ) array.

We thought that this was somewhat redundant, since actually (η,ξ) are good coordinates everywhere but on the saddle point. So we spent a large amount of time trying to

"patch" the coordinates. The idea was to use the chain rule on the saddle and its nearest neighbors, but just carry $\gamma_{\eta\eta}$, $\gamma_{\eta\xi}$ outside of this region. Since the equations are covariant under three-dimensional coordinate transformations, we only have to change the differencing scheme so that it takes ∂_ρ, ∂_z etc. inside the saddle region and ∂_η, ∂_ρ, etc. outside of this region. Unfortunately, it seems that the "noise" introduced on the boundary of the saddle region always causes a numerical instability. We tried both a "small" saddle region, with just nearest neighbors in, and a "large" saddle region containing everything inside $\eta = (NUM+3)*\Delta\eta+\eta_0$, where $\eta = NUM*\Delta\eta+\eta_0$ is the line which intersects the saddle point. Always a "spike" developed on the axis where this boundary occurred. Therefore, we went back to Cadez's approach of using the chain rule over the entire mesh.

To obtain the values of functions on "the other side" of our boundaries we need special commands. The boundaries are: 1) the z-axis ($\rho = 0$ -- note that the part above the hole is $\xi = 0$ while that below the hole to the saddle point is $\xi = \pi/2$), 2) the equatorial plane ($z = 0$, $\xi = \pi/2$), 3) the throat ($\eta = \eta_0$), and 4) the far zone ($\eta = INF*\Delta\eta+\eta_0$). For 1) and 2) we can use the fact that the functions must be either even or odd across the boundary depending on whether they change sign when $\rho \to -\rho$ and $z \to -z$. Obviously, A,B,D are even and C is odd for 1) and 2). As long as we keep $\alpha = 0$ on the throat the metric functions do

not evolve there, and so differencing never comes up.

The wave zone is the most difficult problem. Our solution (although there are probably more sophisticated ones) was to experiment with various differencing schemes on a function satisfying the scalar wave equation and obeying outgoing boundary conditions:

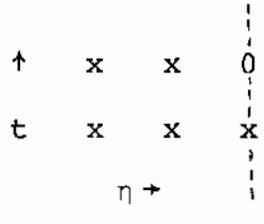
$$(\partial_t^2 - \partial_x^2)\phi = 0 \qquad \phi = \phi(t - x) \quad .$$

We[4] took a line with N points on it, started a Gaussian wave packet moving outward, and let it go "off the end" of the line. In general, one gets a reflection off the boundary which moves inward after the "bounce." By choosing the proper difference scheme:

$$H(I,K+1) = \frac{(V-1)(2V-1)}{(V+1)(2V+1)} [H(I,K) - H(I-2,K+1)]$$

$$+ \frac{2(2V-1)}{V+1} [H(I-1,K) - H(I-1,K+1)] + H(I-2,K) \quad ,$$

we minimize the reflection. Here I ~ η, K ~ t and we used the Heavyside operators to fit the points:



This difference scheme was applied to each ξ = constant ray at the boundary 4) with no coupling between different values of ξ. It is important that this boundary be treated carefully because we will want to measure precisely the radiation as it leaks "over the edge." More work needs to be done here.

Finally, we note that special commands must be used on the axis since $\rho \to 0$ there and terms like ρ^{-1} occur in our equations. Quantities should be finite on the axis, so we suspect any term containing ρ^{-1} will also contain $\partial_\rho H$ terms, where H is even across $\rho = 0$, or terms which are odd across the axis. This is indeed the case, so we are in a situation where we must use L'Hospital's rule for evaluating indeterminant $(\frac{0}{0})$ expressions. For example, if a term in the evolution equations reads $\rho^{-1}C$, we replace it with $\partial\rho C$, or if it reads $\rho^{-1}\partial_\rho A$ we replace it by $\partial_{\rho\rho} A$. Fortunately, it is fairly easy to isolate these terms and put logical checks in to switch to the "axis equations" when the computer finds itself on the axis of rotation.[5] We note also that whenever the equations tell us to take a derivative of $\rho^2 D$ we must factor out the ρ^2 and evaluate it separately, e.g. $\partial\rho(\rho^2 D) = 2\rho D + \rho^2 \partial_\rho D$.

One last point is that on the saddle all first derivatives vanish. The mixed second derivatives of even functions are also zero as are the unmixed second derivatives of odd functions. Since the Jacobian blows up at $(z,\rho) = (0,0)$, we must difference directly here. For that reason we carry in memory the values of (z,ρ) for the 3 nearest neighbors

of the saddle point.

It is important to remember that our grid stays fixed through the entire calculation. The metric components are regarded as functions defined over the grid, which vary in time. At a given time one can plug in the appropriate $(\Delta\eta,\Delta\xi)$ or $(\Delta z,\Delta\rho)$ to a metric component and read off the proper distance between two points on the mesh. For instance, as the holes fall together, our grid lines do not get bunched up between the throat and the saddle; rather the metric function $A = \gamma_{zz}/\psi^4$ will drop from unity towards zero in this region, implying that the proper distance $\psi^2\sqrt{A}\Delta z$ is becoming smaller there.

Let us now turn to the evolution equations themselves. Both Hahn and Lindquist (1964) and Cadez (1971) used the pair of first order in time equations:

$$\partial_0\gamma_{ij} = -2\alpha K_{ij}$$

$$\partial_0 K_{ij} = -2\alpha K_{\ell i}K_j{}^{\ell} + \alpha K K_{ij} + \alpha R_{ij} - \alpha_{.ij}$$

Because we had used the second order equations in slicing one black hole (Estabrook, et al. 1973), we decided to use them for the two black hole problem. Accordingly, we replaced K_{ij} everywhere by $-\frac{1}{2\alpha}\gamma_{ij,0}$. The resulting equations read:

$$\gamma_{ij,00} = \gamma^{\ell m}(\gamma_{\ell i,0}\gamma_{mj,0} - \tfrac{1}{2}\gamma_{\ell m,0}\gamma_{ij,0}) + (\ell n\alpha)_{,0}\gamma_{ij,0} - 2\alpha^2 R_{ij} + 2\alpha\alpha_{.ij} \quad .$$

Again we see the clean separation between "velocity" terms involving only time derivatives and "potential" terms involving only spatial derivatives. The formulas we listed in Chapter IV, which give the Christoffel symbols and curvature tensors of conformally related spaces, are used here to split off the ψ^4 dependence of metric. The Christoffel symbols of the base metric are built out of spatial derivatives of A,B,C,D.

One can see the hyperbolic character of the evolution equations more clearly if we note that we may write out the three dimensional Ricci tensor with a part containing second space derivatives of the metric functions and a part containing only first space derivatives. Neglecting all but the second derivatives one can write, for example:

$$A_{,00} - \alpha^2\psi^{-4}\left[\left(\frac{A}{AB-C^2}\right)(A_{,\rho\rho} - 2C_{,\rho z} + B_{,zz}) + D^{-1}D_{,zz}\right] = \cdots$$

$$B_{,00} - \alpha^2\psi^{-4}\left[\left(\frac{B}{AB-C^2}\right)(A_{,\rho\rho} - 2C_{,\rho z} + B_{,zz}) + D^{-1}D_{,\rho\rho}\right] = \cdots$$

and similarly for C and D. The "effective velocity," for instance:

$$\alpha\psi^{-2}\sqrt{\frac{A}{AB-C^2}}$$

varies over the grid, but approaches unity as we go far from the holes (recall c = 1). To maintain the causal stability

of our difference equations we must choose Δt to be smaller than the minimum of the proper distance across any mesh square. This is equivalent to making the "mesh velocity" $\Delta\eta/\Delta t$ everywhere greater than the "effective" velocity of the metric given above. We therefore scan the mesh every so often to determine if we need to change Δt. As a result, we need to use difference equations for the time derivatives which allow for unequal time steps. These are easily obtained (Allen-1954-page 67) by fitting with a quadratic. If the time steps are h_1 and h_2, then we find that:

$$\begin{array}{l} A_+x \\ \quad\Big\updownarrow h_1 \\ A_0x \\ \quad\Big\updownarrow h_2 \\ A_-x \end{array} \qquad \frac{\partial^2}{\partial t^2} = [\tfrac{1}{2}h_1h_2(h_1+h_2)]^{-1}[h_2A_+ + h_1A_- - (h_1+h_2)A_0]$$

$$\frac{\partial}{\partial t} = \frac{h_2A_+}{h_1(h_1+h_2)} + \frac{(h_1-h_2)A_0}{h_1h_2} - \frac{h_1A_-}{h_2(h_1+h_2)} \quad .$$

which reduces to the usual formulae when $h_1 \to h_2 \to \Delta t$.

Notice that if $\alpha \neq \alpha(t)$ then the terms with covariant derivatives of α change only because of the change in the Christoffel symbols as the metric functions evolve. This means we can use the analytically evaluated spatial derivatives of α and thus keep higher accuracy than we could if $\alpha = \alpha(t)$. We know that α will vary in time for a maximal slicing so that for such slicings we lose accuracy twice: once because we must solve an elliptic equation for α on each sheet and twice because we then have to difference the resulting α numerically.

For the case where α is held fixed in time we need carry only two layers of metric. The second time derivatives are split up so that only A_+ appears on the left hand side of the equation. The first time derivatives are backwards differenced:

$$\partial_t A = \frac{A_0 - A_-}{\Delta t}$$

and then the system is iterated with the first derivatives center differenced. This assures we maintain second order accuracy. In practice, only one or two iterations seem to be necessary. At each point of the mesh, as we sweep across it, the new value A_+ is placed in the lower slot of the A array for that point. It will not be needed for differencing neighboring points. Here again, when we maximally slice, we will have to carry three layers of metric and iterate many more times.

To start this updating procedure from the time-symmetric initial slice, we will need special equations. Of course, all first time derivatives vanish. To get A_+ we note that $A_- = A_+$, so we can make the second time derivative:

$$\partial_t^2 = \frac{2(A_+ - A_0)}{(\Delta t)^2} \qquad (t = 0)$$

Let us turn now to some numerical results. As stated before we will use $\mu_0 = 2$ for initial conditions.

From Table 1, we see that the total mass is M = 1.272a. We will measure time in units of M by inserting $(1.272)^{-1}$M everywhere we find a. To get a rough idea of the evolution we will start with a fairly coarse mesh. The grid with NUM = 3, INT = 14, and INF = 23 is shown in Figure 4. The metric functions A,B,C,D are all 1 or 0 and are shown in Figure 5. One regards the functional value (here flat) as having a copy of the coordinate grid placed on it. Alternatively, this is a perspective drawing[6] of the functions with the height of the function, at a grid point, representing its functional value. After evolving 55 levels we find ourselves 5.96M into the future. By now the metric functions have changed drastically in the strong field region (Figure 6). Of particular importance are the changes in A and C. The scale on these functions is different, but for A the origin of the triad is zero. The plunge A is taking indicates that the metric distance between coordinate lines along the z-axis is decreasing rapidly, and in fact it goes to zero when t = 7.3M. As we shall see shortly, the coordinate lines cross at the saddle point for t = 7.3M. Whether this is also a true physical singularity is not clear yet because I have not programmed the Riemann invariants I and J in yet. However, the elapsed proper time at the saddle point when the lines cross is $\tau = \alpha(0,0)*7.3M = 2.41M$, which is not that far from the $\tau = 3.14M$ required for the Schwarzschild collapse of the throat. Here $\alpha = 0$ on the throats so they do

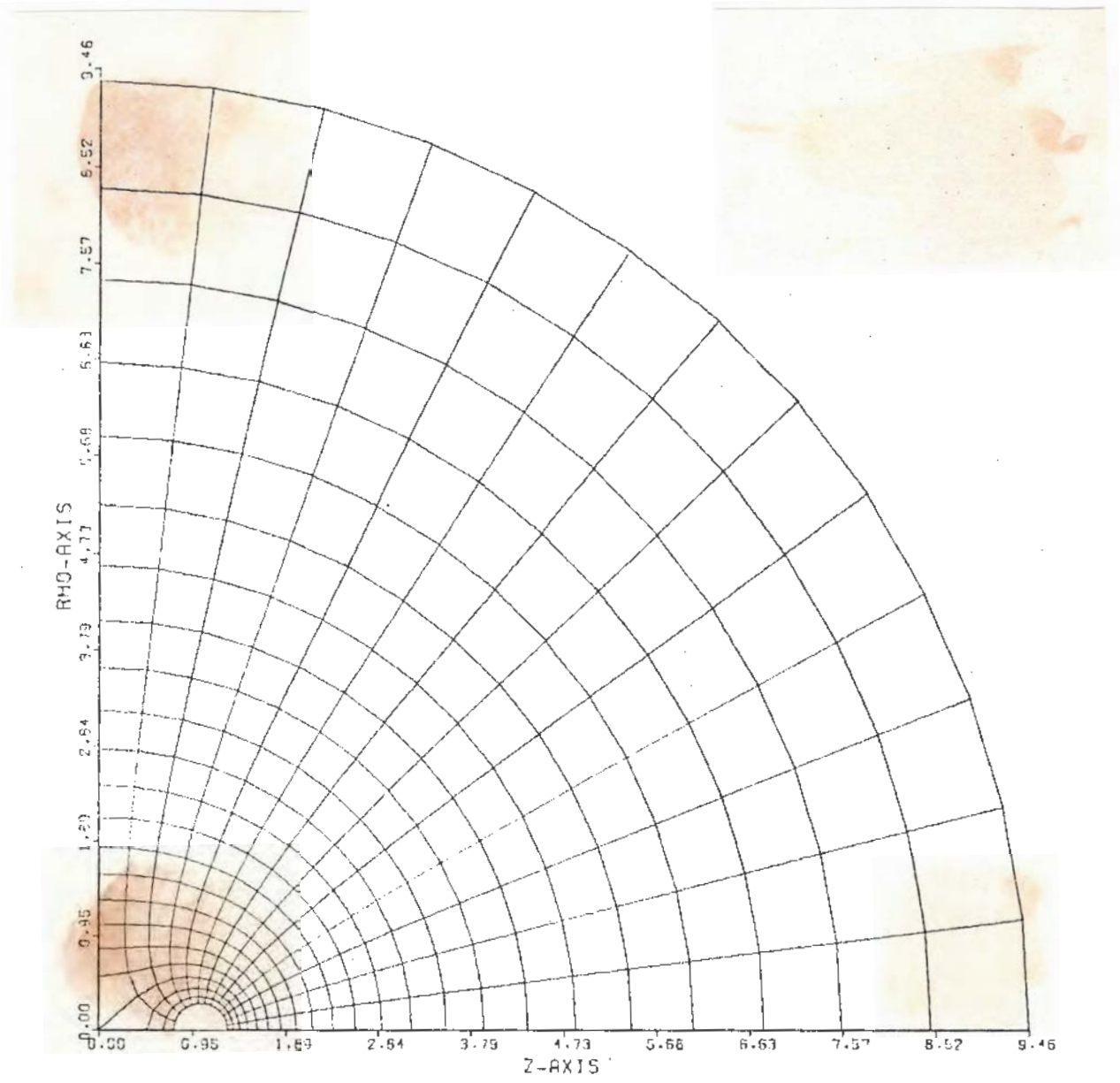


Figure 4

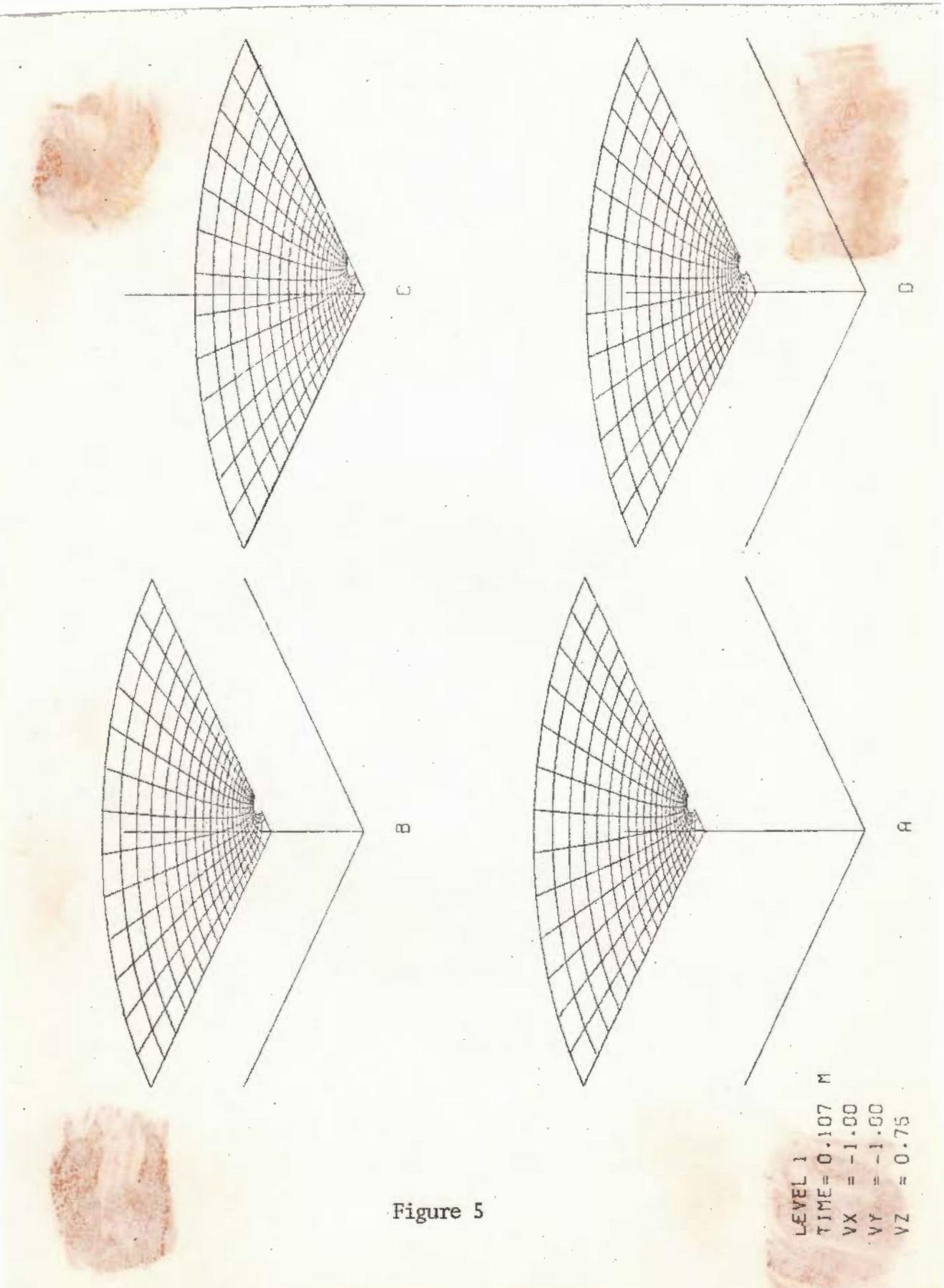


Figure 5

not pinch off, but it may be that the apparent horizon has surrounded the central region and all of it is heading for collapse.

The other critical point is that C is becoming very steep on the axis at about the point where a numerical instability pops up. The fact that C is so steep implies that its first ρ derivative may be getting inaccurate ($C_{\rho\rho} = 0$ on axis).

Moving on to Figure 7 we find the trace of the extrinsic curvature K and the determinant of the conformal 3-metric DET = $(AB - C^2)D$. Because we are not keeping the slices maximal, but only requiring $\Delta\alpha - R\alpha = 0$ on the first slice, K builds up. This means the coordinate lines are converging very strongly near the saddle as we saw by examining A. Notice that the determinant mimics K since we recall that:

$$\partial_t\sqrt{\gamma} = -\alpha K\sqrt{\gamma}$$

and we know that $\sqrt{\gamma} = \rho\psi^6(AB - C^2)D$ and $\dot{\rho} = \dot{\psi} = 0$, so the $\rho\psi^6$ cancels from each side leaving:

$$\partial_t\sqrt{DET} = -\alpha(TRK) \quad .$$

One sees that as $t \to 7.3M$, $A \to 0$, $DET \to 0$, and $K \to \infty$ at the saddle point ($C = 0$ there).

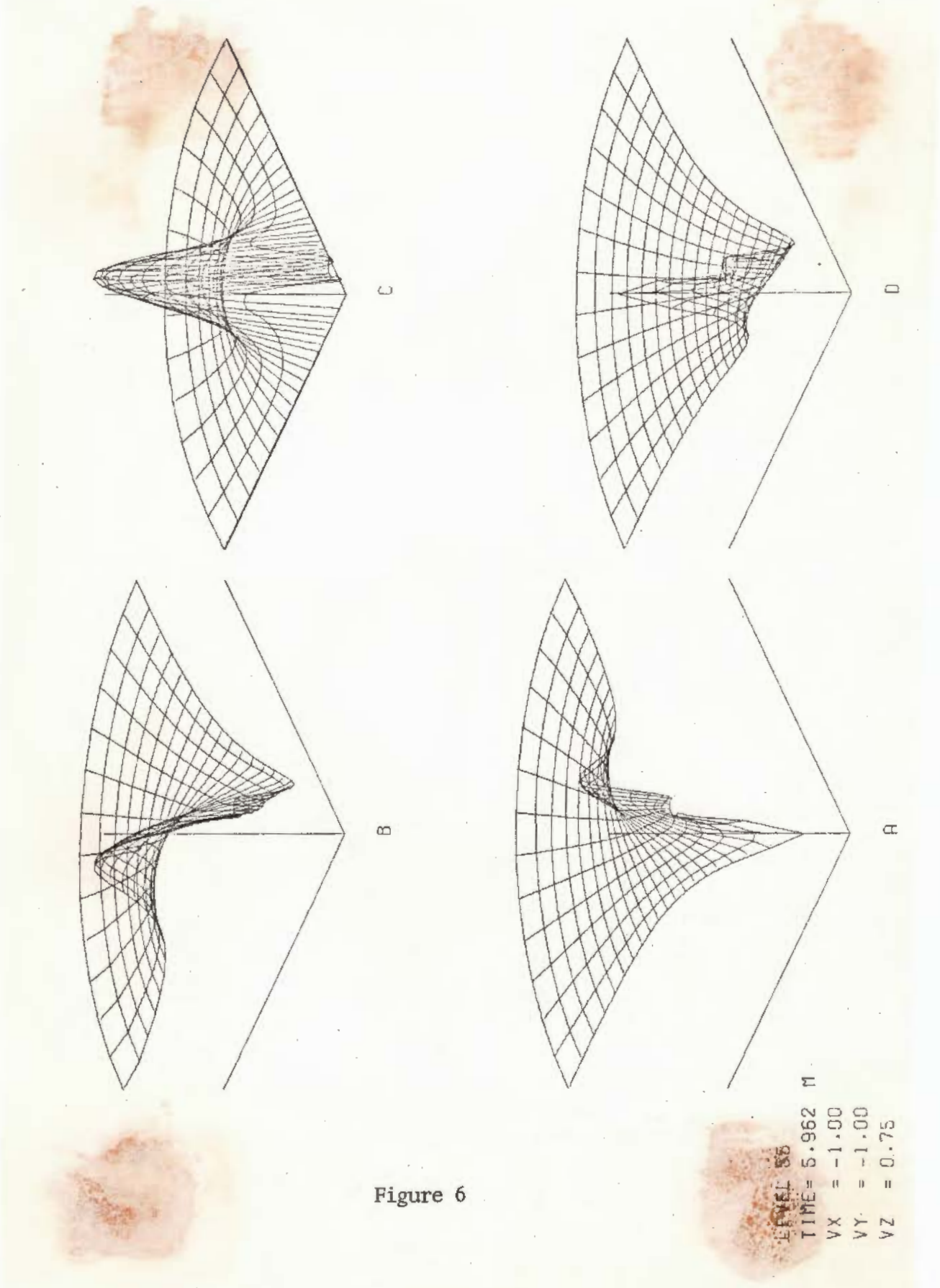


Figure 6

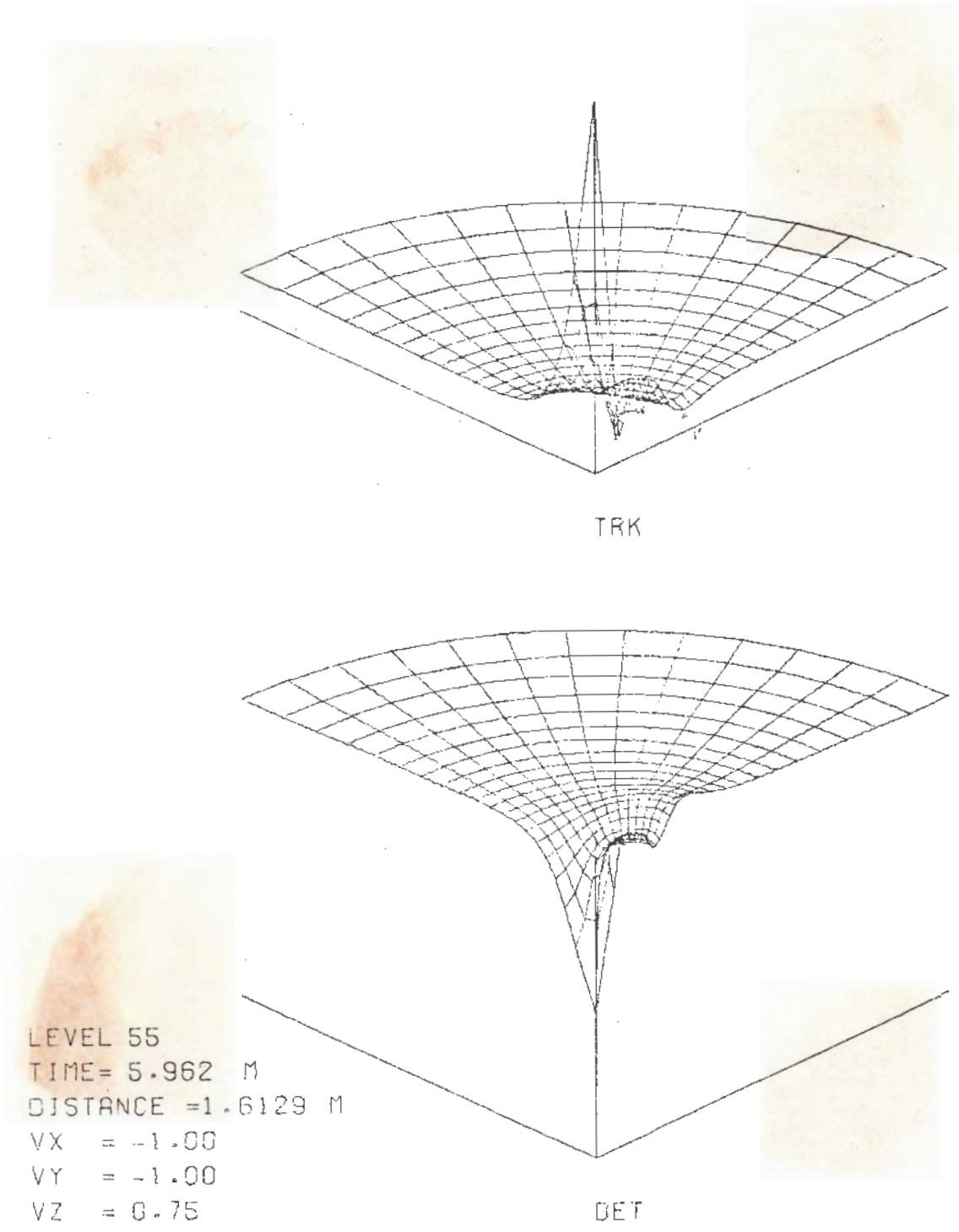

TRK
LEVEL 55
TIME= 5.962 M
DISTANCE =1.6129 M
VX = -1.00
VY = -1.00
VZ = 0.75
DET

Figure 7

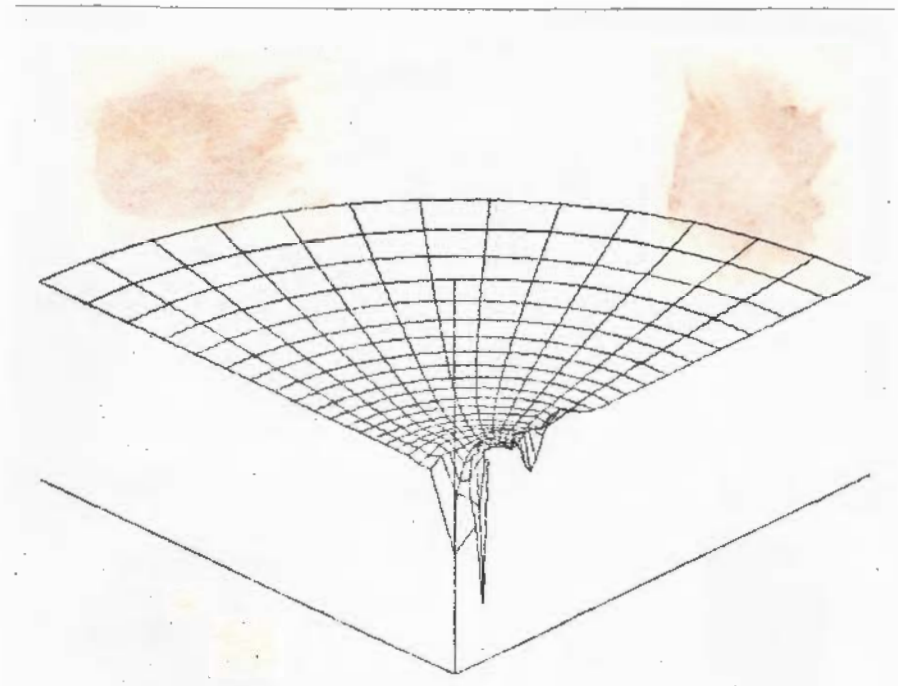
The next Figure 8 gives us two sensitive indicators of error. The Hamiltonian constraint:

$$H = K_{ij}K^{ij} - K^2 - R$$

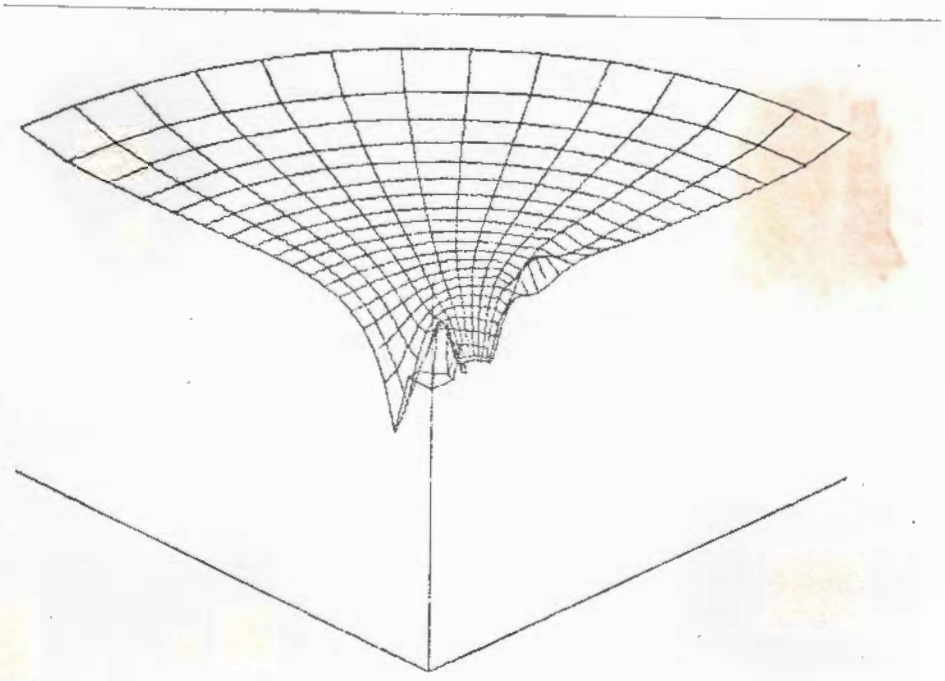
is plotted. Notice that it is zero, as it should be if everything is working right, except at two points. At the saddle and on the axis between the saddle and the throat H is getting out of hand. This is due to the fact that the coarsest part of the grid occurs there, and we are not getting a good approximation to derivatives by our differences. The other bad point is near where I mentioned C was getting so steep. A numerical instability of some sort is developing. Notice that it is <u>not</u> gravitational radiation (wiggles in the curvature) because for that the Hamiltonian constraint would stay zero. We notice that besides this instability there are no wiggles in RIEMANN = ${}^4R_{t\theta\theta t}$. At this point things are not fine tuned enough to tell whether that means there is no radiation being generated. However, as we shall see in a moment, a finer grid does not change this result. Note that the distance from the throat to the saddle point has fallen to 1.61M from the original $\frac{1}{2}\frac{L}{M} = 1.94$.

The next four Figures 9, 10, 11, 12 show the "velocity terms"

$$\gamma^{\ell m}(\gamma_{i\ell,0}\gamma_{jm,0} - \tfrac{1}{2}\gamma_{\ell m,0}\gamma_{ij,0}) \qquad (\dot{\alpha} = 0)$$

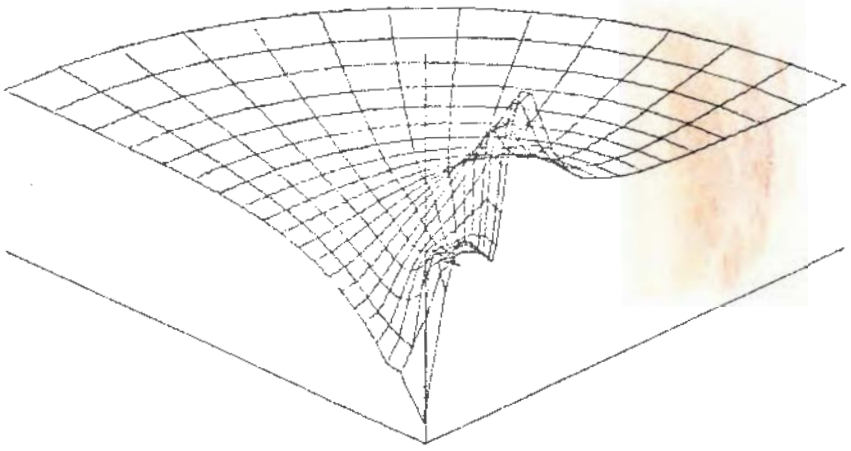


CONSTRAINT

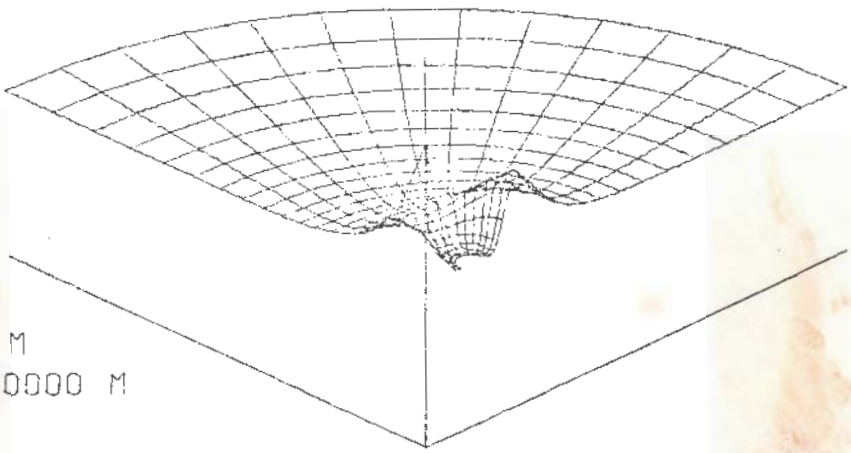


RIEMANN

Figure 8

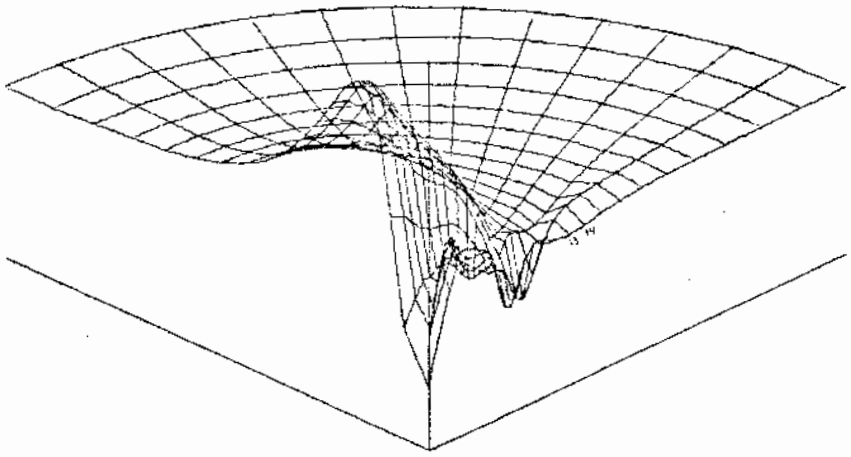


QEE

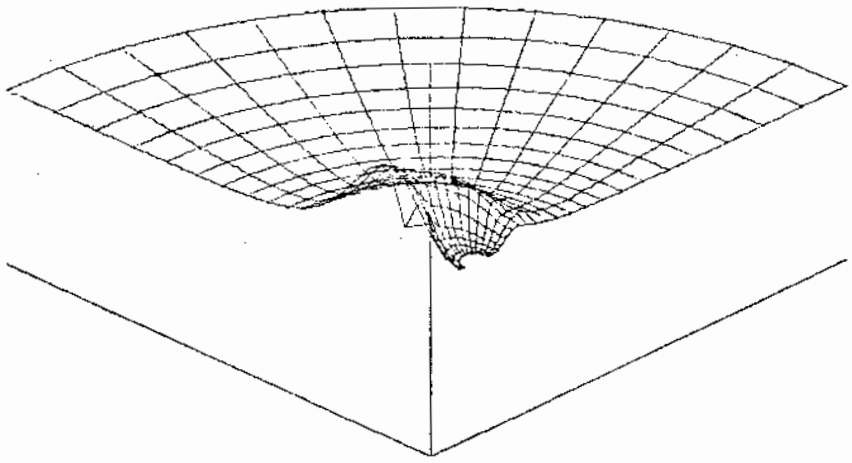


ATT

Figure 9

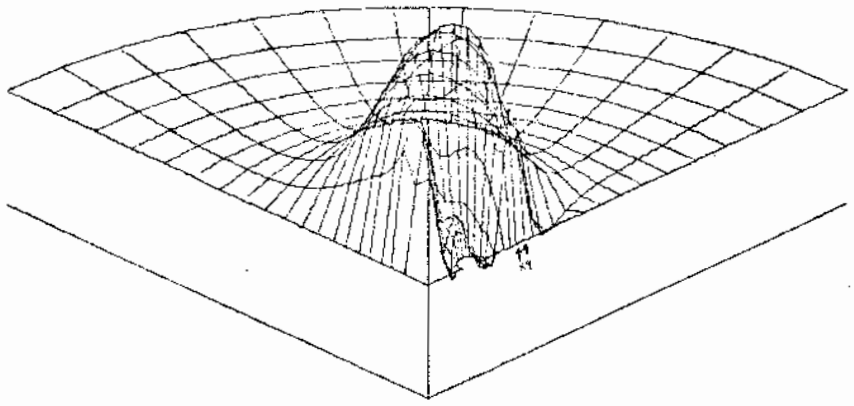


QXX

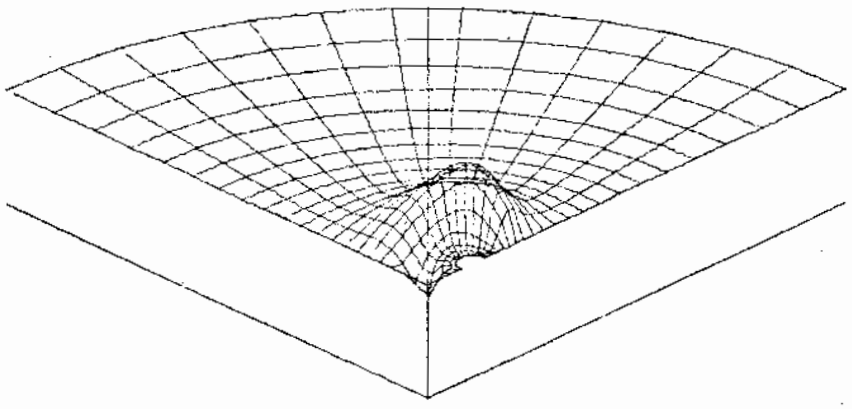


BTT

Figure 10

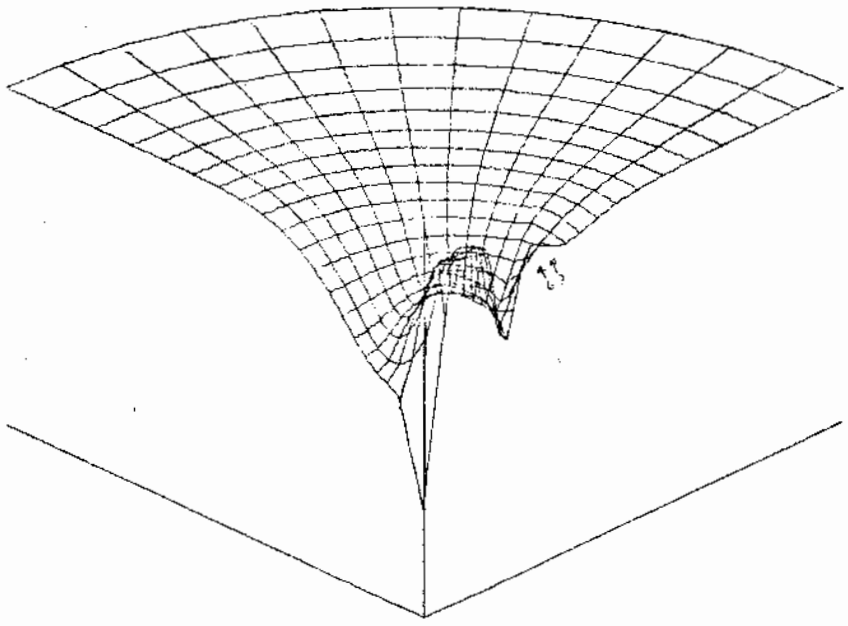


QEX

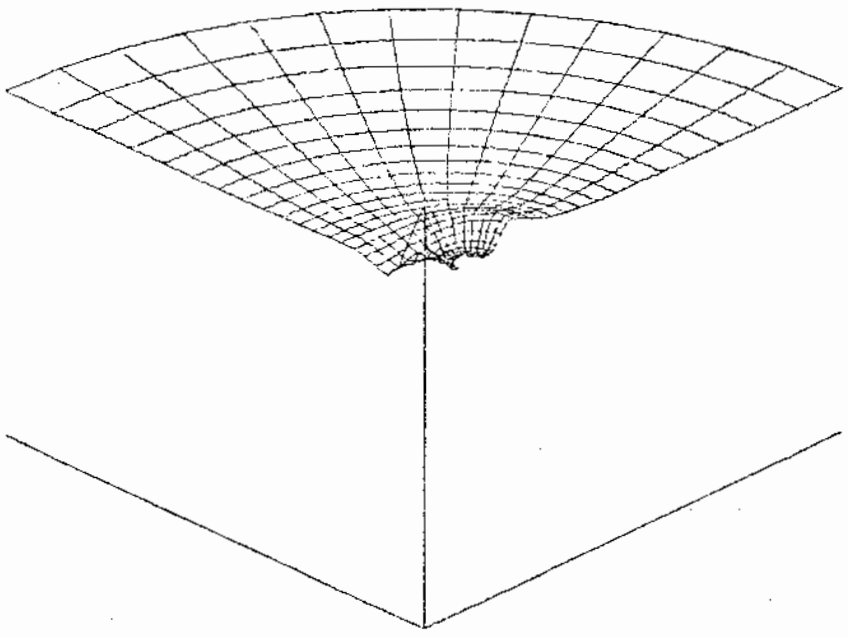


CTT

Figure 11



QPP



QTT

Figure 12

vs. the "potential terms":

$$-2\alpha^2 R_{ij} + 2\alpha\alpha_{.ij}$$

at the time the other Figures are shown. For each metric function (e.g. A) the terms are shown on the same scale so that one can directly compare their strengths. We note that the time derivative terms are becoming quite large, particularly considering that they were zero to start with while the potential terms were not. It is hoped that such an approach can lead to a classification of regions as "velocity dominated" or "normal dominated" (see Eardley 1972 and Liang 1973).

To see the effects of a smaller grid size we made these runs again with about half the grid size. Now NUM = 6, INT = 28, INF = 40. The grid is shown in Figure 13. After 76 levels, the time elapsed is t = 5.58M. Although this is less time than before, we can see from Figure 14 that our coarse grid gave a remarkably good description of TRK and DET. This is useful to know because the coarser the grid we can run on, and still believe the results, the less time and storage is required.

One peculiarity, which has not been resolved, is the numerical instability on the axis. From Figure 15 we see that by halving the mesh size we have made the instability much worse. This is behavior just the opposite of what one might think. Naturally we also halved the time step, and in

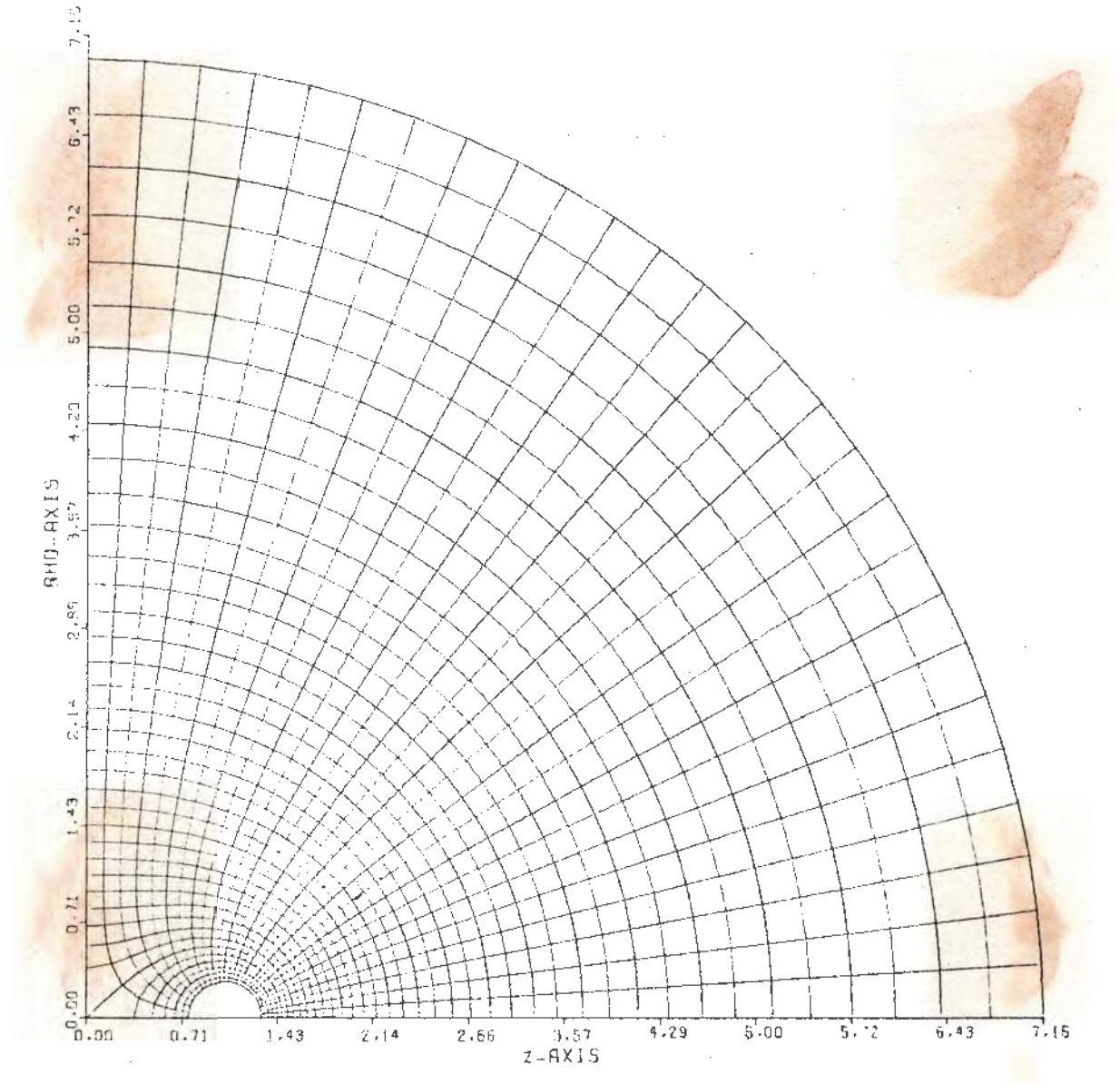


Figure 13

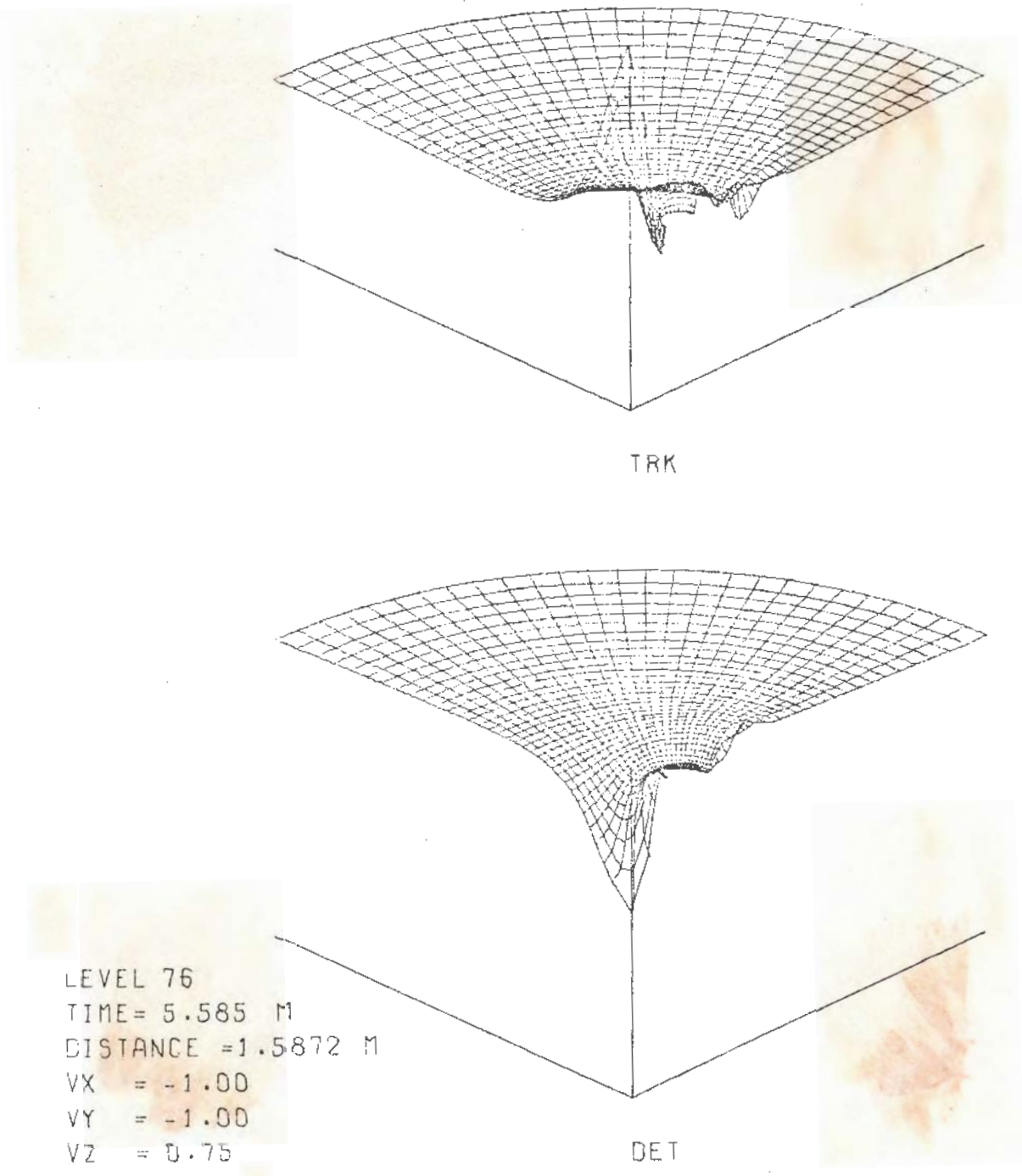
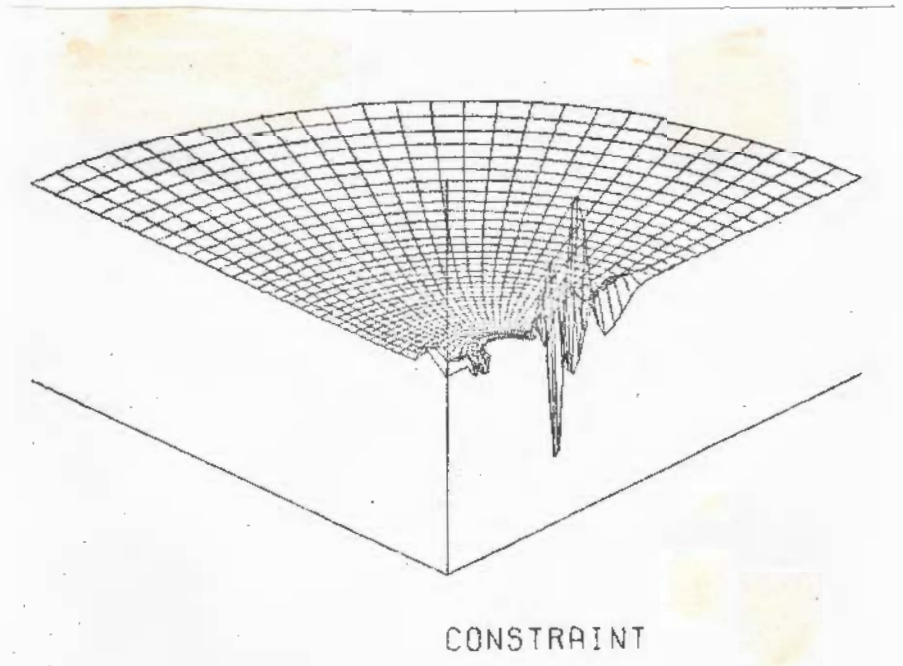

TRK
LEVEL 76
TIME= 5.585 M
DISTANCE =1.5872 M
VX = -1.00
VY = -1.00
VZ = 0.75
DET

Figure 14



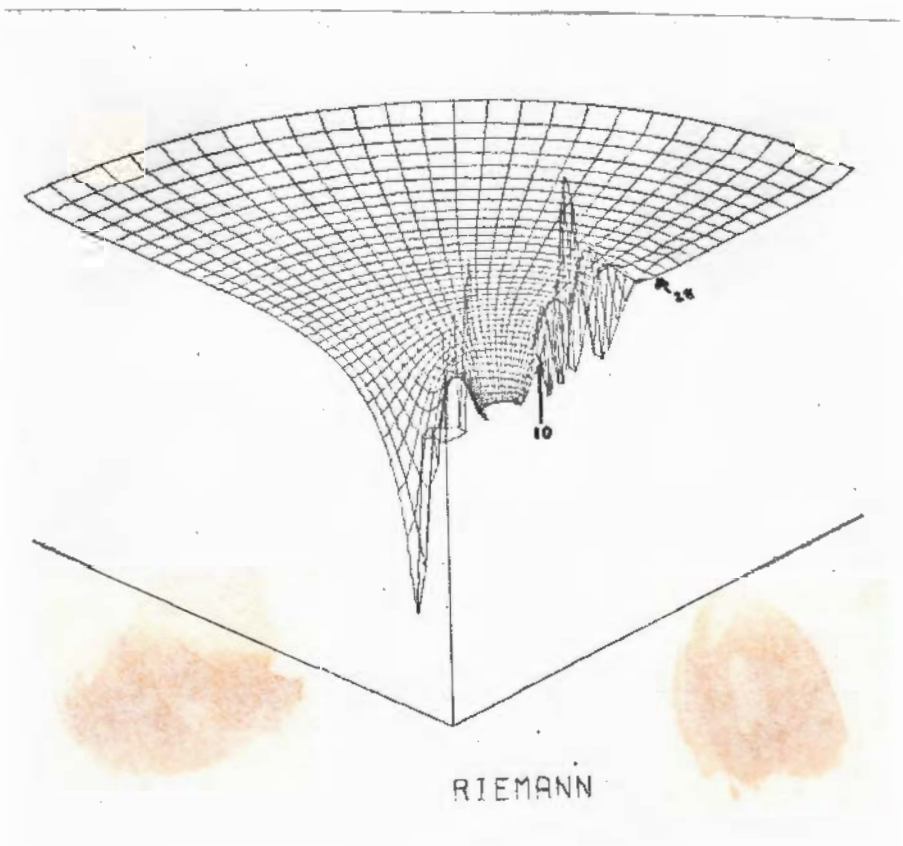


Figure 15

fact I made many runs where I halved and rehalved it with no appreciable change. One good note is that the Hamiltonian constraint has drastically improved in the saddle region. It is actually quite smooth if you think that 1) it is on the coarsest part of the mesh and 2) the quantities which are supposed to "balance" to give zero, namely K, $K_{ab}K^{ab}$ and R, are at their largest values on the saddle.

This last point brings up another important aspect of differencing 2 + 1 partial differential equations: the absolute size of mesh squares. As we remarked earlier, the time step must be chosen smaller than the smallest proper spatial distance across any grid square in order to maintain stability. This is not sufficient, however. The absolute proper spatial distances must also be smaller than the "scale" of variation of the functions we are trying to difference.[7] In our problem the scale is set by M, e.g. the radiation we hope to see will be of wavelength ≃M. Even in this second mesh (Figure 13) the square nearest the saddle point has a spatial proper width of about 0.76M (recall the distance from the throat to the saddle is ~2M). This indicates we might have to use a graded or subdivided net in the saddle region. This also sets a limit on how far out towards the wave zone we can go because the squares get larger as we move out. When they begin to have widths of ≃M we must realize that our expected radiation will not propagate smoothly across it. Finally, we note that if we use a larger μ_0, say $\mu_0 = 3$, the

initial distance from the throat to the saddle will be greater and we will need to put more mesh lines in.

As one last example of an attempt to get rid of the instability on the axis, we show in Figure 16 a mesh offset a half step from the axis. This way one never has to evaluate terms where $\rho \to 0$. No special axis commands are used, and the strict reflection symmetry across the axis is broken. The results, unfortunately, were worse than before. The instability was even larger and more sudden. This instability remains one of the chief unsolved problems we face.

By way of summary, let us compare the extent of spacetime explored in the three attempts to date. Hahn and Lindquist used a mesh 51 x 151 square. They stepped forward 50 time intervals with a final $(t/M) \simeq 0.35$. Cadez used a grid 38 x 71 and after 52 steps was $(t/M) \simeq 1.37$ into the future. The coarse grid we showed was 14 x 23 and with 66 time steps we went $(t/M) \simeq 7.3$. The important number is (t/M) since the number of levels and size of the grid just tells us how fine the mesh is.

The next step is to use maximal slicing. As mentioned before, this is much more complicated. The program has been written and is executing, but it is not completely debugged. To solve for the lapse function which takes us from maximal to maximal we must solve an elliptic equation on each slice. I have used the standard SOR (Simultaneous

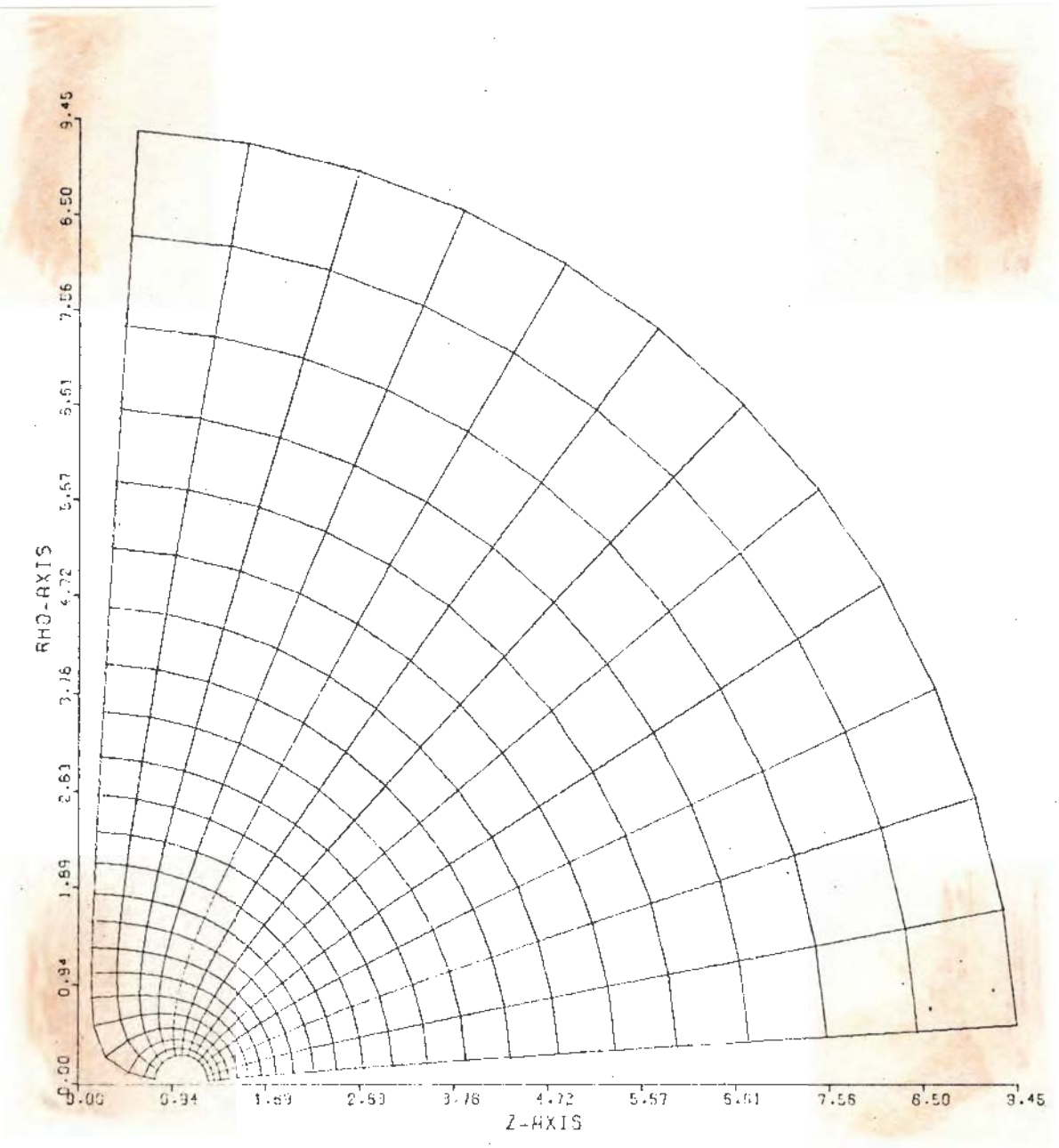


Figure 16

OverRelaxation) technique. (See the books on numerical analysis in the bibliography.) The use of SOR and the choice of the optimum overrelaxation parameter is essential for success. In tests conducted on the initial surface, where we can compare the numerical solution of $\Delta\alpha = 0$ to the analytic solution, I found that after 70 iterations the error per grid point was 10^3 less with the optimum parameter than with the parameter equal to unity (grid size 28 x 40). Further tests are being made to try to determine how many iterations to use per slice to obtain α.

I am sorry to have to close here since things are just getting interesting, but I think the case has been made for the feasibility of using the computer to evolve space-times. What we would like is to get to the point that we can turn this into a laboratory for probing various aspects of the structure of general relativity. Hopefully, we are nearer that goal as a result of this thesis.

References

Chapter VIII

1. I thank Paul Sommers and Richard Matzner for discussions on this point. Roger Penrose (1960) uses a similar argument to show that Schwarzschild, being spherically symmetric, is type {2,2}.
2. One can use the formulae in Gautreau, Hoffman, and Armenti (1972).
3. Our procedure follows the outline of the code developed by Čadež (1971).
4. Steve Christensen carried out these calculations.
5. A powerful test of the accuracy and consistency of these equations is that D$\longrightarrow$ B on the axis. In our calculations, this happens to six or seven significant digits.
6. I thank Tom Criss and Adam Drobot for programming help on these graphic displays.
7. Adam Drobot was very helpful on these points.

Bibliography

This bibliography does not presume to be comprehensive. I have listed articles or books I feel useful to the problems discussed in the text. There are many others I have not included, usually because they are referenced by the articles listed below. I would appreciate the interested reader's calling to my attention any omissions or mistakes. I wish to express my gratitude to my wife, Janet Levarie Smarr, for her essential help in compiling this bibliography.

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