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When Black Holes Collide

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Abstract. The head-on collision of two nonrotating axisymmetric equal mass black holes is treated numerically using Misner's time-symmetric initial data where the two holes are represented by Einstein-Rosen bridges that can be placed arbitrarily distant from one another. We have extracted both $\ell = 2$ and $\ell = 4$ gravitational waveforms resulting from the collision. The normal modes of the final black hole dominate the spectrum in all cases studied. The total energy radiated is computed using several independent methods, and is typically less than $0.002M$.

The coalescence of two black holes is considered to be one of the most promising sources of gravitational waves (Schutz [1993]). In a series of papers (Anninos et al.

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The computations described in this article were carried out on a Cray-YMP (National Center for Supercomputing Applications), a Cray-C90 (Pittsburg Supercomputer Center), a NEC-SX3/44 (Centre de Informatique de Dorval) and a Fujitsu VPX-240 (Calgary High Performance Computing Center). Funding for this project was provided by NCSA and an NSERC (Canada) research grant.

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[1993], [1994] and [1995]) we investigate numerically a special case of the black hole coalescence problem, namely the head-on collision of two equal mass black holes. Building on the work of DeWitt, Cadez, Smarr and Eppley (Smarr et al. [1976] and Smarr [1977] and [1979]) (henceforth abbreviated as DCSE) and the more recent work involving distorted, single black holes (Abrahams et al. [1992], Anninos et al. [1997] and Bernstein et al. [1994]), a 2D code has been developed that is capable of constructing highly accurate numerical solutions to the vacuum Einstein equations.

The method of solution is based upon the 3+1 (or ADM) formalism (Arnowitt et al. [1962]) where the line element can be written as

$$ds^2 = -(\alpha^2 - \beta^i\beta_i)dt^2 + 2\beta_i\, dx^i dt + \gamma_{ij}\, dx^i dx^j. \tag{1}$$

Here α is the lapse function that foliates the four dimensional spacetime into three dimensional spatial hypersurfaces with 3-metric γ_{ij}, and β^i is the shift vector that specifies three dimensional coordinate transformations from time slice to time slice. Greek indices (ranging from 0 to 3) label four dimensional coordinates, and Latin indices (ranging from 1 to 3) label spatial coordinates. Geometric units ($G = c = 1$) are also used.

The vacuum Einstein equations reduce to four constraint equations

$$R - K_{ij}K^{ij} + K^2 = 0, \tag{2}$$

$$\nabla_j(K^{ij} - \gamma^{ij}K) = 0, \tag{3}$$

and twelve evolution equations

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + \nabla_i\beta_j + \nabla_j\beta_i, \tag{4}$$

$$\begin{aligned}\partial_t K_{ij} \;=\; & -\nabla_i\nabla_j\alpha + \alpha\left(R_{ij} + K_{ij}K - 2K_{ik}K^k_j\right) \\ & +\beta^k\nabla_k K_{ij} + K_{kj}\nabla_i\beta^k + K_{ik}\nabla_j\beta^k.\end{aligned} \tag{5}$$

Here R_{ij} is the Ricci tensor formed from the spatial 3-metric γ_{ij}, K_{ij} is the extrinsic curvature, K is the trace of K_{ij} and ∇_j is the covariant derivative with respect to γ_{ij}.

The initial data we consider consists of two throats connecting two isometric sheets. By construction the throats are spheres, centered on the axis along $\rho = 0$, (where (ρ,ϕ,z) are standard cylindrical coordinates) on which boundary conditions relating the metric across the two sheets may be imposed. Since the natural boundaries (the throats and a sphere surrounding the system far from the throats) do not lie along constant (z,ρ) coordinates, it is useful to introduce the "quasi-spherical" Cadez (Smarr et al. [1976]) coordinates (η,ξ) with η being a logarithmic "radial" coordinate and ξ an "angular" coordinate. Cadez coordinates are related to cylindrical coordinates through a complex conformal transformation

$$\chi(\zeta) = \eta + i\xi = \frac{1}{2}\left[\ln(\zeta+\zeta_0) + \ln(\zeta-\zeta_0)\right] + \sum_{n=1}^{\infty} C_n\left(\frac{1}{(\zeta_0+\zeta)^n} + \frac{1}{(\zeta_0-\zeta)^n}\right), \tag{6}$$

where $\zeta = z + i\rho$, and $\zeta_0 = \coth\mu$ is the value of ζ at the throat center. The constant η and ξ coordinate lines of (6) lie along the field and equipotential lines of two equally charged metallic cylinders located at the centers of the two throats $z = \pm\coth\mu$. The coefficients C_n are determined by a least-squares method that set the throats (defined by ${\rho_{th}}^2 + (z_{th} \pm \coth\mu)^2 = 1/\sinh^2\mu$) on an $\eta = \eta_0 =$ constant coordinate line. Both η_0 and the different C_n are computed on a problem

specific basis for different μ using this least-squares procedure. The constant Cadez coordinate lines as viewed in the cylindrical coordinate system are shown in Figure 1a. However the computer views the (η, ξ) coordinates as a rectangular system as shown in Figure 1b.

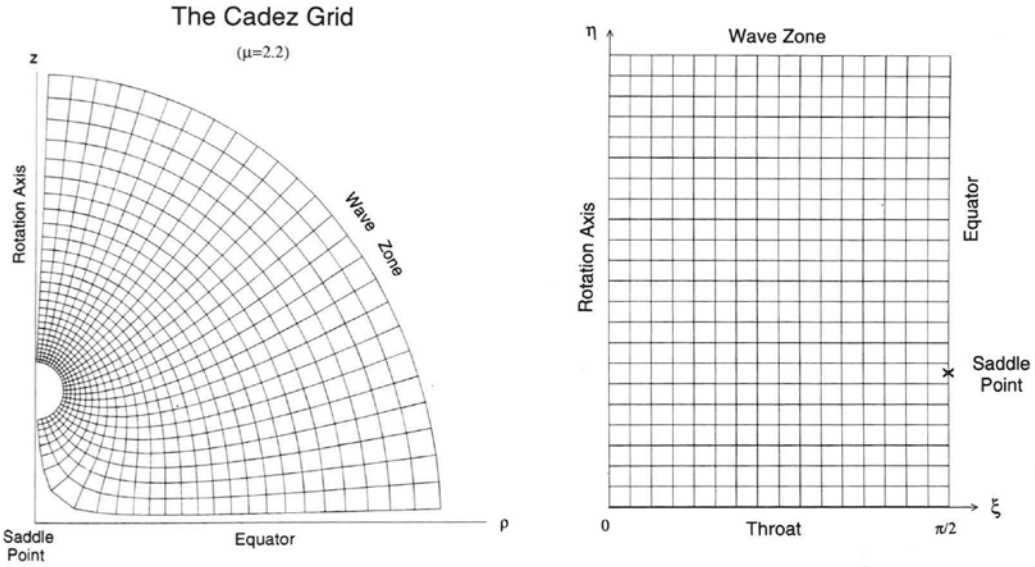


Figure 1 (a) The Cadez grid is constructed for the case $\mu = 2.2$ and displayed in a single quadrant with cylindrical coordinates. The throats are centered on the axis at $z = \pm \coth \mu$. Lines of constant η concentrically surround the throat locally, and become spherical far from the holes. (b) The computational grid is shown in Cadez coordinates.

The most general spacetime four-metric for a non-rotating axisymmetric spacetime may be written as:

$$g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta^i\beta_i & \beta_\eta & \beta_\theta & 0 \\ \beta_\eta & \psi^4 a & \psi^4 c & 0 \\ \beta_\theta & \psi^4 c & \psi^4 b & 0 \\ 0 & 0 & 0 & \psi^4 d \sin^2\theta \end{bmatrix}$$

where (with the exception of $\psi(\eta, \theta)$) the metric coefficients are functions of t, η, and θ.

In 1960 Misner [1960] constructed an analytic time-symmetric solution to the (equal mass) black hole collision initial value problem. The initial 3-metric is conformally flat

$$dl^2 = {\psi_M}^4 \left(d\rho^2 + dz^2 + \rho^2 d\phi^2 \right). \tag{7}$$

The conformal factor ψ_M defined by

$$\psi_M = 1 + \sum_{n=1}^{\infty} \frac{1}{\sinh(n\mu)} \left(\frac{1}{\sqrt{\rho^2 + (z + z_n)^2}} + \frac{1}{\sqrt{\rho^2 + (z - z_n)^2}} \right), \tag{8}$$

(where $z_n = \coth(n\mu)$) solves the Hamiltonian constraint (2) with the proper isometry imposed between the upper and lower sheets. This data set which is axisymmetric, reflection symmetric about the equator ($z = 0$) and time symmetric ($K_{ij} = 0$) represents two equal mass black holes with zero rotation. The two black hole centers are aligned along the axis of symmetry (z-axis) so the physical interaction is a head-on collision along the axis. The free parameter μ is related to the physical

parameters M (half the total ADM mass and approximately the mass of a single black hole when the holes are "infinitely" separated)

$$M = M_{ADM}/2 = 2\sum_{n=1}^{\infty} \frac{1}{\sinh(n\mu)}, \tag{9}$$

and L (the proper distance along the spacelike geodesic connecting the throats)

$$L = 2\left[1 + 2\mu\sum_{n=1}^{\infty} \frac{n}{\sinh(n\mu)}\right]. \tag{10}$$

The effect of increasing μ is to set the two black holes further away from one another and decrease the total mass of the system.

Solutions are constructed using the ADM evolution Equations (4) and (6) written in a second order time explicit finite differencing scheme. The lapse function is determined from the maximal slicing condition ($\mathrm{Tr}K = 0$) and the condition that the 3-metric is diagonal (i.e., $c = 0$) determines the shift vector. Using a method developed by Abrahams and Evans [1990] and applied in Abrahams et al. [1992] to oscillating black hole spacetimes the gravitational wave-forms have been extracted from the numerical values of the metric functions. This method is based on a Regge-Wheeler perturbation expansion and one can construct the the gauge invariant Zerilli function Ψ (Zerilli [1970]). The function Ψ represents the wave-like part of the metric that is radiative at large distances from the source and is commonly used in semi-analytic calculations of black hole normal mode frequencies (Chandrasekhar [1983]). The asymptotic energy flux carried by gravitational waves can be computed from

$$\frac{dE}{dt} = \frac{1}{32\pi}\left(\frac{\partial\Psi}{\partial t}\right)^2, \tag{11}$$

independently for each ℓ mode contribution for the normalization we use for Ψ.

For all of the simulations we have extracted both the $\ell = 2$ and $\ell = 4$ waveforms at radii of 30, 40, 50, 60 and $70M$. By comparing results at each of these radii we are able to check the propagation of waves and the consistency of our energy calculations.

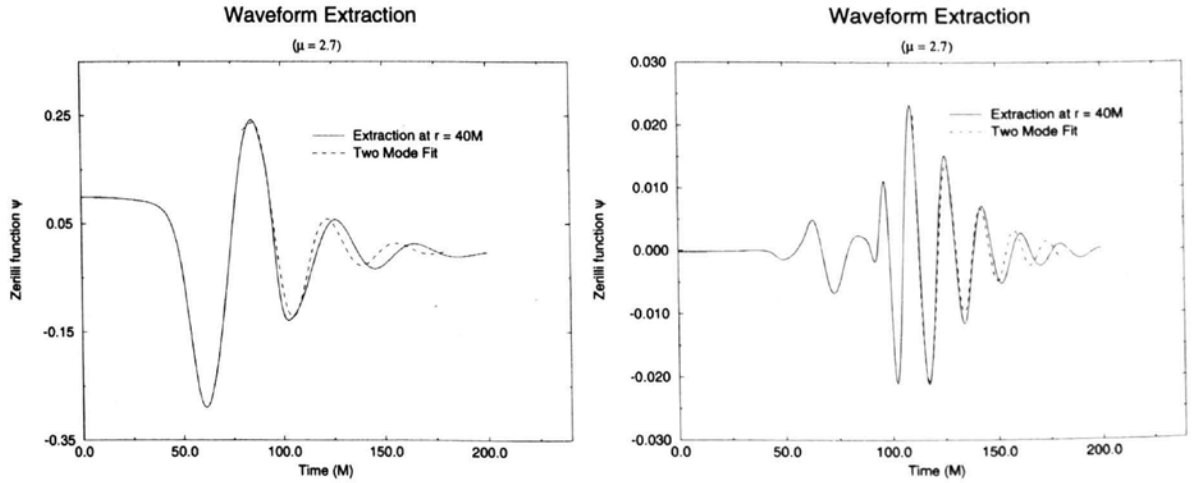


Figure 2 The (a) $\ell = 2$ and (b) $\ell = 4$ waveforms for a black hole collision with $\mu_0 = 2.7$ ($L/M = 12.7$). The solid lines indicate the waveforms measured at $r = 40M$ and the dotted lines are the quasi-normal mode fits to the mass of the final black hole.

Figure 2 shows the waveforms for the $\ell = 2$ and the $\ell = 4$ modes obtained from a single black hole collision. The gravitational waves can be used to measure the mass increase of the single black hole as it absorbs increasing amounts of gravitational wave energy. This explains the difference between the quasi-normal modes and the true wave form.

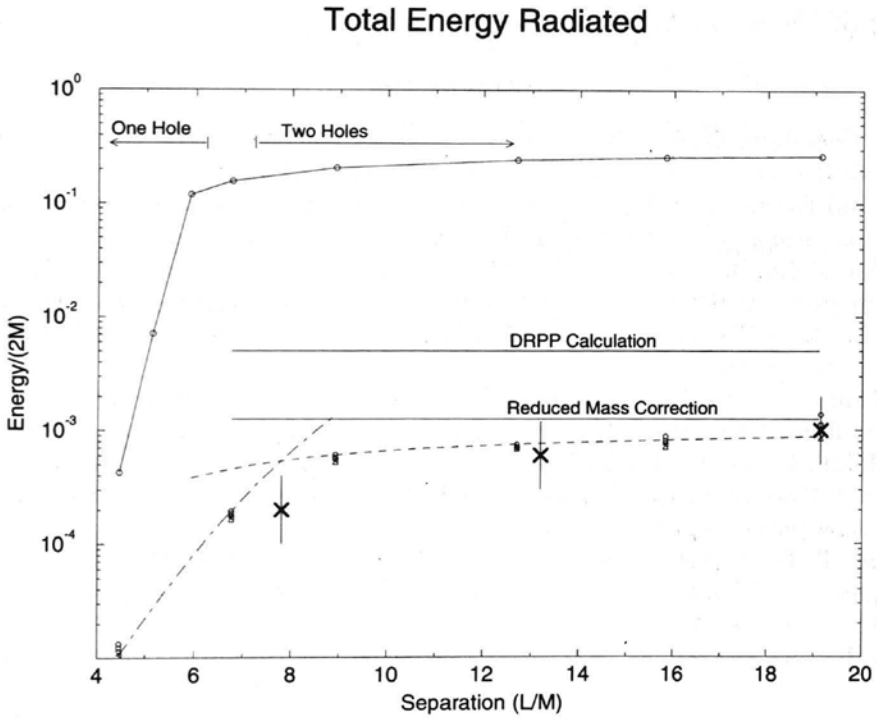


Figure 3 The total gravitational wave energy output is shown for the six parameter studies. The connected circles are the upper limit based on the Hawking area theorem, the clustered symbols show numerical results at various detector locations, and the crosses show early results by Smarr and Eppley with their approximate error bars. The solid line labelled "DRPP Calculation" is the result of a naive application of the point particle result, the solid line labelled "Reduced Mass Correction", takes into account the finite mass of the infalling black hole, and the dashed line is the semi-analytic estimate. Finally, the dot-dashed line is an empirical fit to the data for low values of the separation parameter μ, showing the exponential fall off of the energy in the one black hole regime.

The total radiated energy E can be computed from the Zerilli function using Equation (11). We display these results in Figure 3. The six clusters of unconnected symbols represent six numerical simulations corresponding to the different μ parameter values. Each of the five symbols within a cluster corresponds uniquely to the total integrated $\ell = 2$ energy computed at the five different wave detectors. For reference, the early results of Smarr and Eppley are plotted as large crosses with error bars suggested by Smarr [1979]. Within the large errors quoted, those early results are remarkably consistent with our more accurate results.

Clearly the results in Figure 3 show two distinct regimes, as denoted by the arrows in the upper part of the figure. For $\mu < 1.8$ the initial data contains *one* black hole, and the energy radiated falls off exponentially. We have fit the energy output to an exponential for $\mu \leq 1.8$ and find that the results are approximated well by the formula $E = 3.13 \times 10^{-8} \exp(4.852\mu)$. For $\mu > 1.8$ there are two holes and the energy radiated is somewhat independent of the initial separation. Four lines based on analytic and semi-analytic calculations that treat the system as two

black holes are also shown. The top line is the theoretical upper limit obtained using the Hawking area theorem. The curve labelled "DRPP calculation" is based on Davis et al. [1971] (see also Chandrasekhar [1983]), the result labelled "Reduced Mass Correction" takes into account the finite mass of a black hole falling into its partner (see Anninos et al. [1993]), and the dashed line is the complete semi-analytic calculation discussed in Anninos et al. [1995], accounting for other effects such as tidal heating of the event horizon.

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